

# Inventory Control of Perishable Items in a Two-Echelon Supply Chain

Fariborz Jolai \*<sup>1</sup>, Elmira Gheisariha<sup>1</sup> and Farnaz Nojavan<sup>1</sup>

<sup>1</sup> Department of Industrial Engineering, College of Engineering, University of Tehran, Tehran, Iran

(Received 14 November 2010, Accepted 24 April 2011)

## Abstract

In this paper, we develop an inventory model for perishable items with random lifetime in a two-echelon production-distribution system. There is a manufacturer at the first stage that produces its product with a constant rate. Deterioration in this stage is modeled via a two-parameter Weibull distribution. At the second stage, the retailer places the order and receives the product instantly. The deterioration rate at this stage is a three-parameter Weibull distribution, which its initial value depends on the time the product has spent in the first stage before being transferred. The behavior of different key parameters of the model is analyzed using numerical studies.

**Keywords:** Inventory control, Deteriorating items, Production-distribution systems, Two-echelon supply chains

## Introduction

We develop a mathematical model for a two-echelon supply chain with perishable items. Products are produced in the first stage and are sent to the second stage to be stocked. Different deterioration functions are imposed for two stages. Our purpose is to find the optimal policy, i.e., the optimal production time, production cycle, inventory level and delivery cycle of the system. The wide variety of perishable items and the enormous number of factories and manufacturers dealing with these products as well as the application of two-echelon supply chains for their production and distribution systems is the main motivation of this study.

Integrating production planning and distribution planning in a supply chain is necessary to achieve its optimal performance. These systems are called *Explicit Production-Distribution (EPD)* systems. Among different groups of EPD defined by Chen [1], joint Lot-Sizing and Finished-Product Delivery Problem is the closest to our study.

In this study we assume production period is equal to distribution period and items lose their values in both stages. The deterioration functions are stochastic and follow Weibull distribution. At the first echelon,

deterioration is modeled via a two-parameter Weibull distribution similar to Covert and Philip [3] and at the second echelon deterioration rate follows a three-parameter Weibull distribution as Philip [8].

Studies on integrated inventory models with perishable items when the manufacturers and the retailers coordinate their production and ordering policies have received much attention from researchers in recent years. Yang and Wee [12] considered a two-echelon system with one manufacturer and several customers with constant production rate, deterioration rate and demand. Wee et al. [11] cited two possible flaws in the cost function of Wee and Yang's model and give a proposal to eradicate the flaws. Rau et al. [9] proposed a model similar to that of Yang and Wee [12] with the only difference that the deterioration rate is set to be exponential. Yang and Wee [13] developed a multi-lot-size production and inventory model of deteriorating items with constant production and demand rates. Lo et al [7] derive an optimal solution for an integrated production-inventory model with imperfect production processes and Weibull distribution deterioration under inflation. Cheng and Wee [2] studied a production-

inventory deterioration model considering pricing policy, warranty period, imperfect production and stock dependent demand. Lee and Hsu [6] study the two-warehouse inventory control model for deteriorating items with finite replenishment rate. Wang et al [10] empirically showed different deterioration rates in each echelon affect performances of individuals and integrated inventory policies. To the best of our knowledge, this is the first study considering perishable items in a two-echelon supply chain having production at the first stage and warehouse at the second stage.

The following assumptions are made in developing the mathematical model:

- 1- Demand rate is constant.
- 2- The product is produced on one production line or production machine. There is constant set up cost at the beginning of each production cycle.
- 3- Inventory control is continuous.
- 4- Lead-time is constant and zero
- 5- At the first stage there is only one producer.
- 6- At the second stage there is only one retailer.
- 7- Shortage is not allowed.
- 8- Deterioration of items begins at the first stage right after being produced.
- 9- The rate of deterioration and its parameters are known for both stages.
- 10- Replacing or repairing the deteriorated items is not allowed.
- 11- Our purpose is to find the optimal policy, i.e., the optimal production time, production cycle, inventory level and delivery cycle of the system.

The rest of this paper is organized as follows: In section 2 we develop a mathematical problem based on our assumptions. In Section 3 we solve the proposed model using numerical analysis and investigate the behavior of optimal solution as different parameters of the model change. Conclusion is presented in Section 4.

## Mathematical Model

For describing the model and its solution we need the following notations, see figure 1:

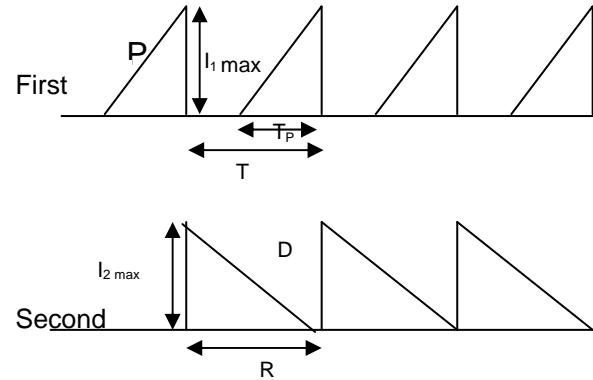


Figure 1: An illustration of the proposed two-echelon system

$D$ : Demand per unit of time

$P$ : Production per unit of time

$T$ : Production cycle (time between two consequent setups)

$R$ : Delivery cycle (time between two consequent deliveries)

$A$ : Delivery cost

$S$ : Set-up cost

$s$ : Set-up time

$C_1$ : Cost of one unit of product for producer

$C_2$ : Cost of one unit of product for the retailer

$h$ : Cost of keeping an item in per unit of time

$T_p$ : Production time in a production cycle

$t$ : time (representing the age of items)

$I_i(t)$ : Inventory level at the  $i^{\text{th}}$  stage ( $i=1,2$ )

$f_i(t)$ : Density function of items life-time at the  $i^{\text{th}}$  stage ( $i=1,2$ )

$F_i(t)$ : Cumulative function of items life-time at the  $i^{\text{th}}$  stage ( $i=1,2$ )

$Z_i(t)$ : Deterioration function of items at the  $i^{\text{th}}$  stage ( $i=1,2$ )

First, we will develop different components of the model separately and at the end, the mathematical model is introduced. We have used the results from Covert and Philip [3] for deterioration rate of product in the first stage when we have a Weibull distribution with two parameters.

$f_1(t)$ : Density function of items life-time in the first stage, has a Weibull distribution with two parameters; its p.d.f. is as follows:

$$f_1(t) = \alpha\beta t^{\beta-1} \exp(-\alpha t^\beta) \quad (1)$$

in which  $\alpha$  is the scale parameter and  $\beta$  is the shape parameter.

$F_1(t)$ : Cumulative function of items life-time in the first stage may be used along with reliability theory to gives us the initial deterioration rate as  $Z'(t) = \frac{f_1(t)}{1 - F_1(t)}$  which

results in:

$$Z'(t) = \frac{\alpha\beta t^{\beta-1} \exp(-\alpha t^\beta)}{\exp(-\alpha t^\beta)} = \alpha\beta t^{\beta-1} \quad (2)$$

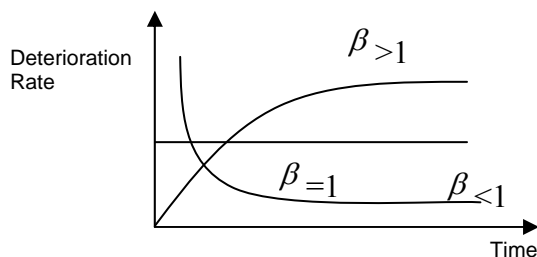
If ( $\beta > 1$ ), the initial deterioration rate is increasing with respect to time.

If ( $\beta < 1$ ), the initial deterioration rate is decreasing with respect to time.

If ( $\beta = 1$ ), the initial deterioration rate is constant and Weibull distribution turns into exponential distribution.

The behavior of the initial deterioration rate,  $Z'(t)$ , as a function of time is shown in the figure 2. Note that  $Z'(t)$  is the deterioration rate for an item in inventory of the first stage at time  $t=0$ . In other words, this function is used when items are in inventory from the beginning of the planning period and we cannot use this function for a producer which produces gradually, i.e., items deteriorate after they are produced and all items are not produced at time  $t=0$ .

Since the production rate is  $P$ , an item which is produced earlier begins to deteriorate earlier, too.



**Figure 2: The behavior of the initial deterioration rate**

We suggest finding a function for deterioration rate by take an average over time, so we will have:

$$Z_1(t) = \frac{\int_0^t \alpha\beta x^{\beta-1} dx}{t} = \alpha t^{\beta-1} \quad (3)$$

According to the fact that items go to the inventory of the first stage immediately after production and their deterioration rate is zero at time  $t=0$ , it implies that  $\beta < 1$ . Comparing a situation in which the warehouse is related to a producer and the items enter the warehouse gradually to a situation in which warehouse is for a retailer who receives an order altogether, we find out that the deterioration rate in the first case is  $1/\beta$  of the deterioration rate in the second stage:

$$Z_1(t) = \frac{1}{\beta} Z'(t).$$

$I_1(t)$ : Inventory level at the first stage

The following differential equation shows the rate of inventory at the first stage:

$$dI_1(t) = -I_1(t)Z_1(t)dt + Pdt$$

In fact the above equation shows that inventory increases with production and decreases with deterioration. By replacing the value of  $Z_1(t) = \alpha t^{\beta-1}$  in the above equation we get:

$$dI_1(t) = Pdt - (I_1(t)\alpha t^{\beta-1})dt$$

$$\frac{dI_1(t)}{dt} + I_1(t)\alpha t^{\beta-1} = P$$

Solving the above equation with standard method results in:

$$I_1(t) = \frac{\int_0^t \exp(\alpha/\beta x^\beta) dx + k}{\exp(\alpha/\beta t^\beta)}$$

To find the constant value of  $k$ , note that the inventory is equal to zero at  $t=0$ :

$$I_1(0) = 0 \Rightarrow k = 0$$

$$\text{So we will have: } I_1(t) = \frac{\int_0^t \exp(\alpha/\beta x^\beta) dx}{\exp(\alpha/\beta t^\beta)}$$

$f_2(t)$ : Density function of item life-time at the second stage.

As mentioned before, the deterioration rate in two-parameter Weibull case begins at time  $t=0$  either with the rate of zero or infinity (see figure 2). Items after being produced and transferred to warehouse begin to deteriorate. While transferring the items to the warehouse of the second stage, deterioration rate is not zero or infinity. In fact it is not reasonable to use two-parameter Weibull distribution. This justifies the use of three-parameter Weibull which gives us more flexibility in modeling the deterioration. We have used the results from Philip [8] for deterioration rate of product in the second stage when we have a Weibull distribution with three parameters. The pdf of this distribution is defined as follows:

$$f_2(t) = \alpha\beta(t-\lambda)^{\beta-1} \exp[\alpha(t-\lambda)^\beta]$$

$\alpha$ : Scale parameter ( $\alpha > 0$ )

$\beta$ : Shape parameter ( $\beta > 0$ )

$\lambda$ : Location parameter ( $\lambda \leq t$ )

$F_2(t)$ : Cumulative distribution function of items life-time in the second stage can be used to find the deterioration rate of the second stage. Similar to the first stage we have:

$$Z_2(t) = \frac{f_2(t)}{1 - F_2(t)}$$

$$F_2(t) = 1 - \exp[-\alpha(t-\lambda)^\beta]$$

This gives us:

$$Z_2(t) = \alpha\beta(t-\lambda)^{\beta-1}$$

In this deterioration function,  $\lambda$  is not constant. Deterioration rate at  $T_p$ , when the production in a cycle of the first stage finishes, is equal to the deterioration rate at

$t=0$  at the second stage. Therefore, we will have:

$$\alpha\beta(0-\lambda)^{\beta-1} = \alpha T_p^{\beta-1}$$

By solving the above equation, we will

$$\text{have: } \lambda = -\frac{T_p}{\beta^{1/\beta-1}}$$

$I_2(t)$ : Inventory level at the second stage

The following differential equation shows the inventory level at the second stage:

$$-dI_2(t) = I_2(t) [\alpha\beta(t-\lambda)^{\beta-1}] dt + Ddt$$

By solving the above equation, we get:

$$I_2(t) = \frac{\int_0^t (-D) \exp[\alpha(x-\lambda)^\beta] dx + k}{\exp[\alpha(t-\lambda)^\beta]}$$

in which  $k$  is a constant and is determined in the following way:

$$I_2(T) = 0 \Rightarrow k = \int_0^T D \exp[\alpha(x-\lambda)^\beta] dx$$

Moreover, by replacing  $k$ , we will have:

$$I_2(t) = \frac{\int_t^T D \exp[\alpha(x-\lambda)^\beta] dx}{\exp[\alpha(t-\lambda)^\beta]}$$

## Objective function

The objective of this model is to minimize the total cost in both stages which has the following components:

Total cost = Setup cost + Holding cost at the first stage + Deterioration cost in the first stage + Delivery cost + Holding cost at the second stage + Deterioration cost at the second stage.

We will compute all parts of the objective function separately and at last, we will add them up to find the objective function as a whole.

*Setup cost*

As mentioned in the problem assumptions, setup cost is supposed to be constant. Since this cost is imposed at the beginning of each production cycle, setup cost per time unit will be:

$$\text{Setup cost per unit of time} = \frac{S}{T}$$

*Holding cost*

In order to compute the holding cost, we should analyze inventory behavior in both stages.

As it is shown in figure 1, setup begins at time zero at the first stage and finishes  $s$  units of time later. At time  $s$ , production begins with a constant rate  $P$  and continues until  $T_p$ . At this time, all of the produced items are transferred to the second stage and with a constant rate  $D$  they are consumed. Note that in the figure 1, deterioration rate is not considered. According to the figure, maximum inventory at the end of production time at the first stage is equal to the maximum inventory at the moment of entering to the warehouse at the second stage. So maximum inventory at the second stage happens to be at  $t=0$  and we have:

$$I_2(t) = I_{\max_2} = \frac{\int D \exp[\alpha(x-\lambda)^\beta] dx}{\exp[\alpha(-\lambda)^\beta]}$$

And because  $I_{\max_1} = I_{\max_2}$ , we can compute holding costs of the first stage

$$\text{as: } \frac{h}{2} * I_{\max_1} * \frac{T_p}{T}.$$

The ratio  $\frac{T_p}{T}$  shows the fraction of time in which the first stage warehouse has inventory. Inventory cost of the second stage can simply be stated as:  $\frac{h}{2} * I_{\max_2}$ .

*Delivery cost*

The delivery cost per unit of time is follows:

$$\text{Delivery cost per unit of time} = \frac{A}{R}$$

*Deterioration cost for the first stage*

Total production at the first stage =  $T_p * P$

Amount of deteriorated items at the first stage =  $T_p * P - I_{\max_1}$

Cost of deterioration at the first stage per unit of time =  $\frac{C_1(T_p * P - I_{\max_1})}{T}$

*Deterioration cost for the second stage*

At the second stage, the amount of deteriorated items is equal to the amount of items that enter the second stage minus the items that are used to fulfill the demand:

Amount of deteriorated items at the second stage =  $I_{\max_2} - DR$ .

Cost of deterioration at the second stage per unit of time =  $\frac{C_2(I_{\max_2} - DR)}{R}$ .

**Model Constraints**

As mentioned before, the maximum inventory at the first stage is when production is finished just before the whole inventory is transferred to the second stage. Maximum inventory at the second stage is when the whole inventory is received from the first stage. So one of the problem constraints is the constraint which shows

$$I_{\max_1} = I_{\max_2}.$$

$$\frac{\int_0^{T_p} P \exp(\alpha/\beta x^\beta) dx}{\exp(\alpha/\beta T_p^\beta)} = \frac{\int_0^T D \exp[\alpha(x-\lambda)^\beta] dx}{\exp[\alpha(-\lambda)^\beta]}$$

Next, note that the production cycle is equal to  $T$ , which consists of setup time, production time and idle time. Second constraint is the one that ensures setup time and production time does not exceed the production cycle.

$$T \geq S + T_p$$

In the above equations, we have:

$$\lambda = -\frac{T_p}{\beta^{1/\beta-1}}$$

Obviously, the model consists of nonlinear non-convex objective function, which has a nonlinear non-convex equality constraint and a nonlinear non-convex inequality constraint. This problem is

difficult to solve using exact methods and in a closed form.

### Numerical Analysis

MATLAB is used and the near-optimal solutions are obtained using a coordinated search method. To implement the search in a narrower space, we can use a simple upper limit and lower limit for the production time:

Clearly, the number of cycles cannot exceed  $\frac{1 - \frac{D}{P}}{s}$ . On the other hand, we have:

$$\text{number of cycles} = T_p \geq \frac{D}{P}.$$

From the above inequalities, we will come to the following conclusion:

$$T_p > \frac{Ds}{(P-D)}$$

For an upper limit on  $T_p$ , it is worthy to note that if  $T_p$  becomes very large, the deterioration rate grows so fast that the inventory level at the first stage does not increase at all. In other word, a very large production time may cause the deterioration rate to be greater than the production rate. Finding the upper limit in this way is very complicated though. Instead we have used the following very simple upper limit in our numerical study:  $T_p < 1-s$ .

In this section, for analyzing the proposed model, we have solved a numerical example and done sensitivity analysis so that we can find out what are the effects of changing different parameters in the model.

#### Example

Assume a system which has a producer at the first stage and a retailer at the second stage. The producer produces an item with the production rate  $P$  and stocks the item at its warehouse. The deterioration of the item at the first stage is following two-parameter Weibull in the form of  $Z_1(t) = \frac{1}{600}t^{0.5}$ . After finishing the production time and transferring items to the warehouse of the second stage, the deterioration function

changes to a three-parameter Weibull of the form  $Z_2(t) = (\frac{1}{600})1.5((t - \lambda)^{0.5})$ .

Other parameters values of the problem are as follows:

$\alpha = 1/600$ ,  $\beta = 1.5$ ,  $h = 0.1$ ,  $C_1 = 4$ ,  $D = 7$ ,  $A = 150$ ,  $S = 100$ ,  $s = 2$ ,  $C_2 = 4$

The optimal policy of the system is obtained as follows:

Optimal production and delivery cycle ( $T=R$ )=171600

Production time=64308

Maximum inventory=1272277

Total cost=9106

Next we show the impact of different assumptions and parameters on the key elements of the optimal policy.

1- The influence of the assumption that items are perishable on total cost, maximum inventory level,  $I_{max}$ , and optimal production cycle,  $T_p$ , is summarized in table 1.

As it is expected, adding the assumption of perishability to the problem assumptions will result in increasing the total cost. The behavior of the two variables  $T_p$  and  $I_{max}$  with and without perishable items are depicted in figures 3 and 4 respectively.

By adding perishability assumption to the model, increasing  $T_p$  results in  $I_{max}$  being increasing up to some point and then declining. This is because by increasing the inventory, deterioration rate increases and after some time, it becomes larger than the production rate.

The behavior of  $T_p$  and  $T$  are exactly the same as  $T_p$  and  $I_{max}$ .

	Item is not perishable	Item is perishable	Rate of change
Total Cost	7755.90	9106	17.40
Maximum Inventory	157.16	127.22	-19.05
Optimal Production Cycle	22.56	17.16	-23.93

Table 1: The effect of perishable items

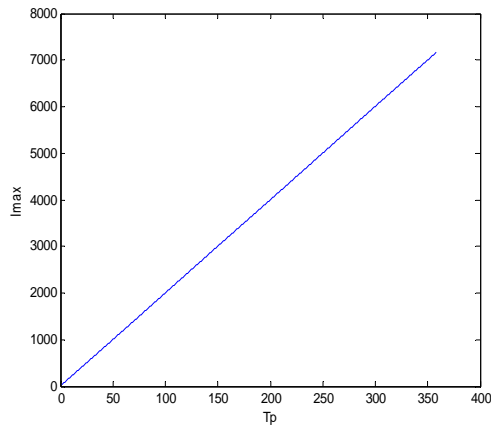


Figure 3: Behavior when item is not perishable

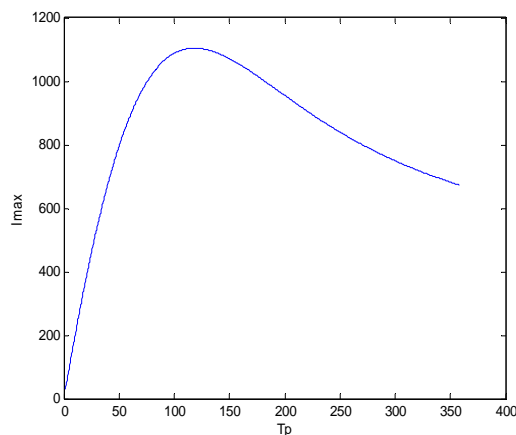


Figure 4: Behavior when item is perishable

2- Changing in  $\alpha$  in the interval  $(0, 1/300)$  will result in the following changes in the optimal solution:

By increasing  $\alpha$ , the rate of deterioration will increase too, so it is expected that total system cost increases too (see figure 5). By increasing  $\alpha$  we predict that  $I_{max}$  decreases, because increasing  $\alpha$  makes the rate of deterioration to increase. As a result, the inventory should decrease. This implies that the production cycle decreases too (see figure 6).

3- The changes in the holding cost in the interval of  $(0, 0.1)$  will result in the following changes in the optimal solution:

By increasing the holding cost, we predict  $I_{max}$  would decrease because increasing the holding cost, implies the inventory level to decline.

The result of comparing the optimal points with respect to holding cost changes is summarized in the table 2.

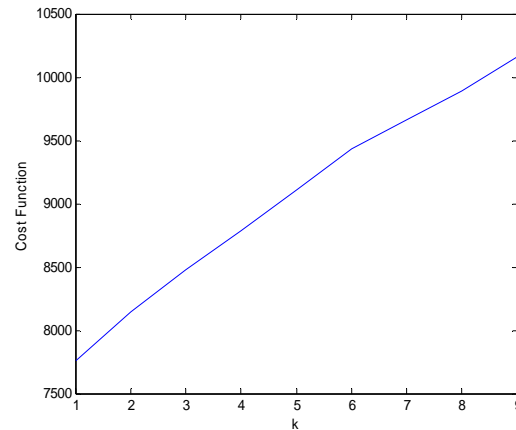


Figure 5: The influence of  $\alpha$  on total cost function

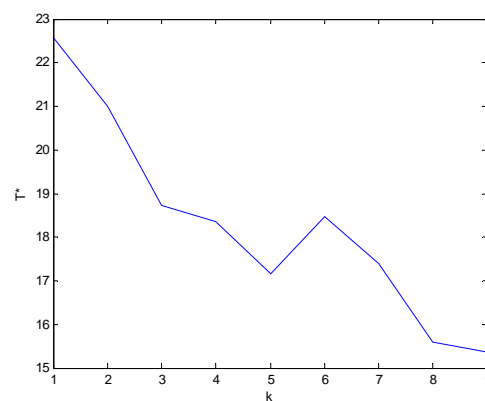


Figure 6: The influence of  $\alpha$  on  $T_p$

The deterioration cost and setup time did not show any influence on TC,  $T_p$  and  $I_{max}$ .

As we can see, the model is more sensitive to  $\alpha$  and holding cost but it is not sensitive to other parameters. This means that using non perishable models for the case of perishable items may lead to the significant errors in total cost estimation as well as mistake managerial decisions such as quantity order. Also these experiments showed that having precise values of  $\alpha$  and holding cost are very important from point of view of management.

## Conclusion

In this paper we have developed a two-echelon inventory model for perishable items with stochastic life-time. At the first stage, the deterioration rate is following a two-parameter Weibull distribution while at the second stage, it is modeled via a three-parameter Weibull distribution.

The optimal solution is obtained using numerical methods and the influence of different model parameters on the optimal policy is considered. The results reveal that a small change in the deterioration parameters has a great effect in the total cost. Our goal is to provide a general framework for these kinds of problems and illustrate how this complicated system can be modeled. To the best of our knowledge, this is the first paper which considers two different deterioration rates for two stages with a high level of flexibility in modeling.

Inventory cost	Total Cost	Maximum Inventory	Optimal Production Cycle
%100	1.19	99.31	13.68
%75	1.13	99.31	13.68
%50	1.06	120.27	16.32
%25	0.99	120.27	16.32
.	0.91	127.22	17.16
%-25	0.83	147.99	19.68
%-50	0.73	147.99	19.68
%-75	0.63	189.06	24.36
%-100	0.50	216.06	27.24

**Table 2: The influence of changes in holding cost**

## References:

- 1- Chen, Z.-L. (2004). "Integrated Production and Distribution Operations: Taxonomy, Models, and Review." Handbook of Quantitative Supply Chain Analysis: Modeling in the E-Business Era, Simchi-Levi, D., Wu, S.D., Shen, Z.-J., (Eds), Kluwer Academic Publishers.
- 2- Chung, C.J. and Wee, H-M. (2008). "An integrated production-inventory deteriorating model for pricing policy considering imperfect production, inspection planning and warranty-period and stock-level-dependant demand." *International Journal of Systems Science*, Vol. 38, PP. 823-837.
- 3- Covert, R. and Philip G. (1973). "An EOQ Model for Items with Weibull Distribution Deterioration." *AIIE Trans.* Vol. 5, PP. 323-326.
- 4- Hill, R.M. (1997). "The Single-Vendor Single-Buyer Integrated Production-Inventory Model with a Generalized Policy." *European Journal of Operational Research*. Vol. 97, PP. 493-499.
- 5- Hill, R.M. (1999). "The Optimal Production and Shipment Policy for the Single-Vendor Single-Buyer Integrated Production-Inventory Problem." *International Journal of Production Research*. Vol. 37, PP. 2463-2475.
- 6- Lee, C.C. and Hsu S-L (2009). "A two-warehouse production model for deteriorating inventory items with time-dependent demands." *European Journal of Operational Research*. Vol. 194, PP. 700-710.
- 7- Loa, S-T, Wee, H-M. and Huang, W-C. (2007). "An integrated production-inventory model with imperfect production processes and Weibull distribution deterioration under inflation." *Int. J. Production Economics*. Vol. 106, PP. 248-260.
- 8- Philip, G.C. (1974). "A Generalized EOQ Model for Items with Weibull Distribution Deterioration." *AIIE Trans* . Vol. 6, PP. 159-162.
- 9- Rau, H., Wu, M. and Wee, H. (2003). "Integrated Inventory Model for Deteriorating Items under a Multi-Echelon Supply Chain Environment." *Int. J. Production Economics*. Vol. 86, PP. 155-168.
- 10- Wang, K-J, Lin, Y.S. and Yu, J C.P. (2011). "Optimizing inventory policy for products with time-sensitive deteriorating rates in a multi-echelon supply chain." *Int. J. Production Economics*. Vol. 130 , PP. 66-76.
- 11- Wee, H.M., Jong, J.F. and Jiang, J.C. (2007). "A Note on a Single-Vendor and Multiple-Buyers Production-Inventory Policy for a Deteriorating Item." *European Journal of Operational Research*. Vol. 180, PP. 1130-1134.



- 12- Yang, P.C. and Wee, H. M. (2002). "A Single-Vendor and Multiple-Buyers Production–Inventory Policy for a Deteriorating Item." *European Journal of Operational Research*. Vol. 143, PP. 570–581.
- 13- Yang, P-C. and Wee, H-M. (2003). "An integrated multi-lot-size production inventory model for deteriorating item." *Computers & Operations Research*. Vol. 30, PP. 671–682.
-