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RESEARCH PAPER

On the derivation of the distribution of the overshoot and undershoot stochastic process in increasing Lévy Processes: a renewal theory approach

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Abstract

Lévy processes with increasing sample paths or subordinators are widely used in Operations Research and Engineering. The main areas of applications of these stochastic processes are insurance mathematics, inventory control, maintenance and reliability theory. Special and well-known instances of these increasing processes are stationary Poisson and compound Poisson processes. Since increasing Lévy processes are mostly regarded as special instances of continuous time martingales the main properties of Lévy processes are derived by applying general results available for martingales. However, understanding the theory of martingales requires a deep insight into the theory of stochastic processes and so it might be difficult to understand the proofs of the main properties of increasing Lévy processes. Therefore, the main purpose of this study is to relate increasing Lévy processes to simpler stochastic processes and give simpler proofs of the main properties. Fortunately, there is a natural way linking increasing Lévy processes to random processes occurring within renewal theory. Using this (sample path) approach and applying properties of random processes occurring within renewal theory we are able to analyze the undershoot and overshoot random process of an increasing Lévy process. Next to well-known results we also derive new results in this paper. In particular, we extend Lorden's inequality for the renewal function and the residual life process to the expected overshoot of an increasing Lévy process at level r.

Keywords:
Overshoot Stochastic Process, Undershoot Stochastic Process, Lévy Process, Limit Distribution.

Introduction

The theory of Lévy processes (stochastic processes with stationary and independent increments) are a branch of modern probability theory and cover a large class of well-known stochastic processes such as Poisson processes, compound Poisson processes and Brownian motion. These processes are named after the French mathematician Paul Lévy who played a crucial role in developing the theory of these processes. The main contributions to the theory of Lévy processes were made between 1930 to 1940s by Paul Lévy, Alexander Khintchine, Kiyosi Ito and Bruno de Finetti. Most of these results and extensions are discussed in Bertoin (1996), Kyprianou (2006), Sato (1999) and Applebaum (2009). Since these books cover the general theory of Lévy processes with real increments and this theory is strongly related to the theory

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of continuous time martingales understanding these books require a mature knowledge of martingale theory and are difficult to understand for researchers outside the field of stochastic processes.

After these processes were introduced and the main theory was developed it became clear that these processes can be served as important building blocks in applications in the field of finance, engineering and physics. Examples of applications to finance and physics are discussed in Barndorf-Nielsen et al. (2001). In particular in finance, we mention that Lévy processes are used for modelling asset returns using the so-called exponential Lévy models (Cont and Tankov (2004)). Lévy processes are also used to model degradation processes in reliability theory (Abdel-Hameed (2014), Kahle et al. (2016)) and storage models (Abdel-Hameed (2014), Kyprianou (2006)). For a recent extensive overview within degradation processes the reader is referred to Li et al. (2020). In the application to reliability and storage models it is assumed that these Lévy processes have increasing sample paths (also called subordinators). Degradation processes in reliability theory measure the accumulative input of damage to a system and are therefore increasing by definition as are the input processes in storage models measuring the accumulative input over time. In reliability theory it is assumed that the system will be replaced if the degradation process reaches a certain threshold damage value. Therefore, it is important to know at which time such an event will happen and how much the overshoot is over this threshold value. In our study, we consider the class of Lévy processes having increasing sample paths and apply a different approach to verify these limiting results approximating a continuous time increasing Lévy process by a renewal process and making use of limit results for renewal processes.

The sole and only purpose of this study is to derive the distribution and limit distribution of the overshoot and undershoot stochastic process for increasing Lévy processes by means of a technique more easily understandable for industrial engineers. As already mentioned, these results are already known for a long time (Bertoin (1996), Kyprianou (2006)). In this paper we use the natural relationship between renewal theory and increasing Lévy processes also observed in Bertoin et al. (1999). In Bertoin et al. (1999) the expression of the limit theorems occurring for the overshoot and undershoot process in renewal theory were used to conjecture similar expressions for limit distributions for the overshoot and undershoot stochastic process in increasing Lévy processes. To verify these conjectures the compensation formula for predictable processes (see page 7 of Bertoin (1996)) was used. As quoted from their paper they say in the third line of Section 4 using their listing of lemmas “this is not trivial, since a direct proof would involve the interchanging of two limit operations: taking the level x to infinity as in Lemma 1 and taking the time interval a to zero as in Corollary 1”. In this note we give an easy and natural way to apply this discretization approach justifying this change of limit operations. This approach, requiring a longer proof, avoids the use of the more complicated compensation formula for predictable processes and makes only use of more elementary techniques. At the same time this discretization approach is also a natural approach in simulating a sample path of a Lévy process having increasing sample paths (Cont and Tankov (2004)).

The outline of this study is as follows. In Section 2 the relevant theory of Lévy processes with increasing sample paths is discussed. We first start in Section 2.1 with the basic definition of a Lévy process with increasing sample paths and its relation to renewal processes. In particular, we discuss the relation of the overshoot and undershoot random process and the hitting time of level r of a Lévy process having increasing sample paths with the same random variables of this Lévy process only observed at equidistant points in time. Moreover, in this section we mention all the known results from renewal theory which are of use to prove similar results for a Lévy process. In Section 2 the asymptotic behavior of the hitting time of level r is derived relating it to the asymptotic behavior of the renewal function. In the same section we present a proof of the cdf and limiting cdf of the overshoot and undershoot random variables using the same renewal theory approximation approach. All of these results are known using a more complicated proof technique. We end this study with a summary and directions for future research.

Overshoot and undershoot in increasing Lévy processes and their connection to the age and residual life stochastic processes in renewal theory.

We start this section with the definition of a Lévy process having increasing sample paths. In the remainder of this paper such a process is called an increasing Lévy process.

Definition 2.1. A stochastic process $\boldsymbol{X} = \{\boldsymbol{X}(t) : t \geq 0\}$ on a given probability space $\boldsymbol{X} = \{\boldsymbol{X}(t) : t \geq 0\}$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is called an increasing Lévy process if
(1) For ever $\epsilon > 0$ it follows that $\lim_{t\to s} P(|\boldsymbol{X}(t) - \boldsymbol{X}(s)| > \epsilon) = 0$ or equivalently the stochastic process $\boldsymbol{X}$ is continuous in probability.
(2) $\boldsymbol{X}(0) = 0$ and $\boldsymbol{X}(t) \geq 0$ for every $t > 0$.
(3) The process $\boldsymbol{X}$ has independent and stationary increments.

A stochastic process is called a Lévy process if Condition 2 in Definition 2.1 is replaced by $\boldsymbol{X}(0) = 0$. It is shown on page 21 of Protter (1992) that there exists a unique modification of a Lévy process $\boldsymbol{X}$ which is a càdlàg process. A stochastic process is called a càdlàg process if its sample paths are right continuous having left-hand limits. In the remainder of this paper, we will use this modification and additionally assume that $\mu_1 := E(\boldsymbol{X}(1))$ is finite and positive. It is easy to show using the independent increments property of Lévy processes that the Fourier transform of the random variable $\boldsymbol{X}(t)$ for every $t > 0$ satisfies

$$\mathbb{E}(e^{iu\boldsymbol{X}(t)}) = \mathbb{E}(e^{iu\boldsymbol{X}(1)})^t \tag{1}$$

for every $u \in \mathbb{R}$. Also, for any Lévy process having a finite moment μ_1 it is easy to show using the independent stationary increments of a Lévy process that

$$\mathbb{E}(\boldsymbol{X}(t)) = t\mu_1. \tag{2}$$

If the Lévy process has a second finite moment $\mu_2 := \mathbb{E}(\boldsymbol{X}^2(1))$ then the variance $\sigma^2(\boldsymbol{X}(t))$ of the random variable $\boldsymbol{X}(t)$ is finite for every $t > 0$. Again, by the independent stationary increments property it is easy to show for every $t > 0$ that

$$\sigma^2(\boldsymbol{X}(t)) = t\sigma^2(\boldsymbol{X}(1)). \tag{3}$$

Since an increasing Lévy process has increasing sample paths it is possible to apply a discretization approach considering the realizations of an increasing Lévy process at equidistant points. At these equidistant points the increasing Lévy process can be regarded as a renewal process. Using this relation between these two classes of stochastic processes we present a simplified approach obtaining the cdf and limiting cdf of the overshoot and undershoot stochastic process for increasing Lévy processes.

Also using this discretization approach, we derive known results for the asymptotic behavior for the expected hitting time at level r of such a process. To present this approach, we first introduce some notation. Let the random variable $\boldsymbol{T}(r)$ denote the hitting time at level $r > 0$ of an increasing Lévy process $\boldsymbol{X}$ defined by

$$\boldsymbol{T}(r) := inf\{t \geq 0 : \boldsymbol{X}(t) > r\}. \tag{4}$$

For any $m \in \mathbb{N}$ we consider now the sampled discrete time increasing Lévy process $\boldsymbol{X}_m =$

$\{\boldsymbol{X}(n2^{-m}) : n \in \mathbb{Z}_+\}$ and this process describes the increasing Lévy process $\boldsymbol{X}$ observed at the times $n2^{-m}$, $n \in \mathbb{Z}_+$. The hitting time $\boldsymbol{T}_m(r)$ of level $r > 0$ for the sampled version $\boldsymbol{X}_m$ of this process is then defined by

$$\boldsymbol{T}_m(r) := 2^{-m} inf\{n \in \mathbb{Z}_+ : \boldsymbol{X}(n2^{-m}) > r\}. \quad (5)$$

To relate the random variable $\boldsymbol{T}_m(r)$ to the random variable $\boldsymbol{T}(r)$ we observe by the monotonicity of the sample paths and relation (5) that for every $m \in \mathbb{N}$ and $r > 0$

$$\boldsymbol{T}_m(r) - 2^{-m} = 2^{-m} sup\{n \in \mathbb{Z}_+ : \boldsymbol{X}(n2^{-m}) \leq r\} \leq \sup\{t > 0 : \boldsymbol{X}(t) \leq r\} \leq \boldsymbol{T}(r) \quad (6)$$

and

$$\boldsymbol{T}_m(r) \geq inf\{t > 0 : \boldsymbol{X}(t) > r\} = \boldsymbol{T}(r). \quad (7)$$

Hence, we obtain that

$$\boldsymbol{T}_m(r) - 2^{-m} \leq \boldsymbol{T}(r) \leq \boldsymbol{T}_m(r) \quad (a.s) \quad (8)$$

for every $m \in \mathbb{N}$ and $r > 0$. The abbreviation a.s means almost surely with respect to the probability measure $\mathbb{P}$. Also, it is easy to check for every $r > 0$ that the sequence of random variables $\boldsymbol{T}_m(r)$, $m \in \mathbb{N}$ are decreasing in m and by relation (8) we obtain

$$\lim_{m\uparrow\infty} \boldsymbol{T}_m(r) \downarrow \boldsymbol{T}(r) \quad (a.s). \quad (9)$$

Note that any increasing Lévy process is a jump process (cf. Protter (1992)). We now define the overshoot stochastic process $\boldsymbol{W} = \{\boldsymbol{W}(r) : r > 0\}$ of an increasing Lévy process $\boldsymbol{X}$ by

$$\boldsymbol{W}(r) := \boldsymbol{X}(\boldsymbol{T}(r)) - r \quad (10)$$

and the undershoot stochastic process $\boldsymbol{V} = \{\boldsymbol{V}(r) : r > 0\}$ by

$$\boldsymbol{V}(r) := r - \boldsymbol{X}(\boldsymbol{T}(r)^-). \quad (11)$$

with $\boldsymbol{T}_m(r)^- := \lim_{s\downarrow 0} \boldsymbol{T}_m(r) - s$. For every $m \in \mathbb{N}$ the overshoot stochastic process $\boldsymbol{W}_m = \{\boldsymbol{W}_m(r) : r > 0\}$ of the sample version $\boldsymbol{X}_m$ is defined by

$$\boldsymbol{W}_m(r) := \boldsymbol{X}(\boldsymbol{T}_m(r)) - r, \quad (12)$$

and the undershoot stochastic process $\boldsymbol{V}_m = \{\boldsymbol{V}_m(r) : r > 0\}$ of the same sampled version by

$$\boldsymbol{V}_m(r) := r - \boldsymbol{X}(\boldsymbol{T}_m(r) - 2^{-m}). \quad (13)$$

Since the stochastic process $\boldsymbol{X}$ is a càdlàg process having increasing sample paths it follows by relations (9) and (10) and the sequence of random variables $\boldsymbol{T}_m(r), m \in \mathbb{N}$ is decreasing in m that

$$\lim_{m\uparrow\infty} \boldsymbol{W}_m(r) \downarrow \boldsymbol{W}(r) \quad (a.s). \quad (14)$$

For fixed $m \in \mathbb{N}$ the random variable $\boldsymbol{W}_m(r)$ can be easily interpreted as the residual life random variable evaluated at time r of a renewal process $\boldsymbol{N}_m = \{\boldsymbol{N}_m(t) : t \geq 0\}$ having independent and identically distributed interarrival times $\boldsymbol{Y}_{km} := \boldsymbol{X}(k2^{-m}) - \boldsymbol{X}((k-1)2^{-m}), k \in \mathbb{N}$. In particular, it is easy to verify using relation (5) and (12) that

$$\boldsymbol{T}_m(r) = 2^{-m}(\boldsymbol{N}_m(r) + 1) \quad (a.s) \quad (15)$$

and

$$r + \boldsymbol{W}_m(r) = \sum_{k=1}^{\boldsymbol{N}_m(r)+1} \boldsymbol{Y}_{km} \quad (a.s). \tag{16}$$

Similarly, the random variables $\boldsymbol{V}_m(r)$ defined in relation (13) can be seen as the age random variable evaluated at time r of the same renewal process $\boldsymbol{N}_m$. By relation (8) we obtain using the monotonicity of the sample paths that for every $r > 0$ and $m \in \mathbb{N}$

$$r - \boldsymbol{X}(\boldsymbol{T}(r) - 2^{-m}) \leq \boldsymbol{V}_m(r) < \boldsymbol{V}(r) \quad (a.s). \tag{17}$$

Since it is easy to verify that the sequence of random variables $\boldsymbol{T}_m(r) - 2^{-m}, m \in \mathbb{N}$ is increasing it follows by relations (17) and (11) using again the monotonicity of the sample paths that

$$\lim_{m\uparrow\infty} \boldsymbol{V}_m(r) \uparrow \boldsymbol{V}(r) \quad (a.s). \tag{18}$$

Before we study in the next section the undershoot, overshoot and hitting time variables in an increasing Lévy process, we show the following dominance result between the random variables $\boldsymbol{W}_m(r)$ and $\boldsymbol{W}(r)$. The notation $\sim$ denotes equality in distribution and so $\boldsymbol{X}\sim\boldsymbol{Y}$ means that the random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same cdf.

Lemma 2.2. For every increasing Lévy process $\boldsymbol{X}$ on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ it follows for every $m \in \mathbb{N}$ and $r > 0$ that

$$\boldsymbol{W}(r) \leq \boldsymbol{W}_m(r) \leq \boldsymbol{W}(r) + \boldsymbol{X}(\boldsymbol{T}(r) + 2^{-m}) - \boldsymbol{X}(\boldsymbol{T}(r)) \; (a.s) \tag{19}$$

with the random variable $\boldsymbol{X}(\boldsymbol{T}(r) + 2^{-m}) - \boldsymbol{X}(\boldsymbol{T}(r))$ independent of the random variable $\boldsymbol{W}(r)$ and

$$\boldsymbol{X}(\boldsymbol{T}(r) + 2^{-m}) - \boldsymbol{X}(\boldsymbol{T}(r)) \sim \boldsymbol{X}(2^{-m}). \tag{20}$$

Proof. Using relation (8) and the increasing Lévy process $\boldsymbol{X}$ has increasing sample paths it follows for every $m \in \mathbb{N}$ that

$$\boldsymbol{W}(r) \leq \boldsymbol{X}\big(\boldsymbol{T}_m(r)\big) - r = \boldsymbol{W}_m(r) \; (a.s). \tag{21}$$

To show the other inequality in relation (19), we observe again by the monotonicity of the sample paths and relation (8) that

$$\boldsymbol{W}_m(r) = \boldsymbol{X}\big(\boldsymbol{T}_m(r)\big) - r \leq \boldsymbol{X}(\boldsymbol{T}(r) + 2^{-m}) - r \; (a.s). \tag{22}$$

Since $\boldsymbol{T}(r)$ is a $\mathcal{F}_t$-stopping time of the Lévy process $\boldsymbol{X}$ and each Lévy process renews itself at a finite stopping time (see theorem 32 of Protter (1992)) it follows that

$$\boldsymbol{X}(\boldsymbol{T}(r) + h) - r = \boldsymbol{W}(r) + \boldsymbol{X}(\boldsymbol{T}(r) + h) - \boldsymbol{X}\big(\boldsymbol{T}(r)\big) \sim \boldsymbol{W}(r) + \boldsymbol{Y}(h) \tag{23}$$

with $\boldsymbol{Y}(h) \sim \boldsymbol{X}(h)$ independent of the random variable $\boldsymbol{W}(r)$. Applying relations (22) and (23) we obtain the upper bound in relation (19) and the result in relation (20).□

It is easy to see using the interpretation of the age process and the residual life process in renewal theory that for every $r > x$ and every $m \in \mathbb{N}$ it follows

$$\{\boldsymbol{V}_m(r) > x\} = \{\boldsymbol{W}_m(r - x) > x\} \tag{24}$$

and in general, for every w > 0 and r > x > 0

$$\{\boldsymbol{V}_m(r) > x, \boldsymbol{W}_m(r) > w\} = \{\boldsymbol{W}_m(r-x) > x + w\}. \quad (25)$$

Applying relation (8), we will analyze in Section 2 the behavior of the function $r \to \mathbb{E}(\boldsymbol{T}(r))$ relating it to the behavior of the expectation of the random variable $\boldsymbol{T}_m(r)$. Also applying relations (14), (18) and Lemma 2.2 we will determine in that section the cdf and limiting cdf of the random variable $\boldsymbol{W}(r)$ and $\boldsymbol{V}(r)$ and the random vector $(\boldsymbol{V}(r), \boldsymbol{W}(r))$using well known results from renewal theory (Çınlar (1975)). These limiting results in renewal theory are part of the curriculum taught to Industrial Engineers. Before discussing these results in renewal theory, we introduce the following definition taken from Feller (1971).

Definition 2.3. A cdf F of a nonnegative random variable $\boldsymbol{X}$ is called arithmetic if it is concentrated on the set of points $n\lambda, n \in \mathbb{Z}_+$ for some $\lambda > 0$. The largest $\lambda > 0$ having this property is called the span of the cdf F. The cdf F is called non-arithmetic, if there does not exist some $\lambda > 0$ satisfying the above property

It is easy to see that any continuous cdf satisfying $F(0^+) := lim_{t\downarrow 0} F(t) = 0$ of a nonnegative random variable $\boldsymbol{X}$ satisfies $\mathbb{P}(\boldsymbol{X} = x) = 0$ for any $x > 0$ and so such a cdf is non-arithmetic. We now mention the following well-known results in renewal theory. We denote by F_t the cdf of the random variable $\boldsymbol{X}(t)$.

Lemma 2.4. If $\boldsymbol{X}$ is an increasing Lévy process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the cdf F_1 is non-arithmetic and has a finite positive first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then for every $m \in \mathbb{N}$ and $r > 0$

$$\lim_{r\uparrow\infty} \mathbb{E}\big(\boldsymbol{T}_m(r+h) - \boldsymbol{T}_m(r)\big) = \frac{h}{\mu_1}. \quad (26)$$

If the second moment $\mu_2 = \mathbb{E}(\boldsymbol{X}^2(1))$ is finite, then

$$\lim_{r\uparrow\infty} \mathbb{E}\big(\boldsymbol{T}_m(r)\big) - \frac{r}{\mu_1} = \frac{\sigma^2\big(\boldsymbol{X}(1)\big)}{2\mu_1^2} + 2^{-(m-1)}. \quad (27)$$

and for every r > 0

$$0 \leq \mathbb{E}\big(\boldsymbol{T}_m(r)\big) - \frac{r}{\mu_1} \leq \frac{\sigma^2\big(\boldsymbol{X}(1)\big)}{\mu_1^2} + 2^{-m}. \quad (28)$$

Proof. Using relation (1) we obtain applying Lemma 3 in Chapter 15.1 of Feller (1971) or Corollary 3.63 of Kawata (1972) that the cdf F_1 is non-arithmetic implies the cdf $F_{2^{-m}}$is non-arithmetic for every $m \in \mathbb{N}$. Since $\mathbb{E}\big(\boldsymbol{X}(2^{-m})\big) = 2^{-m}\mu_1$ the result in relation (26) follows using relation (15) and applying the key renewal theorem (see Theorem 5.2 of Karlin and Taylor (1975)) to the non-arithmetic renewal process $\boldsymbol{N}_m$. To show relation (27) we observe applying again the key renewal theory (see page 197 of Karlin and Taylor (1975)) and relation (2) that

$$\lim_{r\uparrow\infty} \mathbb{E}\big(\boldsymbol{N}_m(r)\big) - \frac{r2^m}{\mu_1} = \frac{\sigma^2\big(\boldsymbol{X}(2^{-m})\big)}{2\mathbb{E}\big(\boldsymbol{X}(2^{-m})\big)^2} - \frac{1}{2}. \quad (29)$$

By relation (2) and (3) we know that

$$\frac{\sigma^2\big(\boldsymbol{X}(2^{-m})\big)}{2\mathbb{E}\big(\boldsymbol{X}(2^{-m})\big)^2} = \frac{2^m\sigma^2\big(\boldsymbol{X}(1)\big)}{2\mu_1^2}. \quad (30)$$

Applying now relation (15) we obtain from relations (29) and (30) the result in relation (27). To show relation (28) we observe by Lordens inequality (Lorden (1970)) for renewal functions that for every $r > 0$ (for a simplified proof of this inequality see Lemma 2.3 of Frenk et al. (1997))

$$0 \leq \mathbb{E}(\boldsymbol{N}_m(r)) + 1 - \frac{r2^m}{\mu_1} \leq \frac{\sigma^2(\boldsymbol{X}(2^{-m}))}{2\mathbb{E}(\boldsymbol{X}(2^{-m}))^2} + 1.$$

This shows the desired result applying the same arguments as in part 2.□

Observe Lordens inequality also holds for arithmetic distributions and so we can delete the condition in Lemma 2.4 that the cdf F_1 is non-arithmetic. Another result needed from renewal theory is given by the following. This result is well known and an immediate consequence of Wald's identity for stopping times (Karlin and Taylor (1975)).

Lemma 2.5. If $\boldsymbol{X}$ is an increasing Lévy process on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and a finite positive first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then for every $m \in \mathbb{N}$ and $r > 0$ it follows that

$$\mathbb{E}(\boldsymbol{W}_m(r)) = \mu_1 \mathbb{E}(\boldsymbol{T}_m(r)) - r. \tag{31}$$

Proof. Since the random variable $\boldsymbol{N}_m(r) + 1$ is a stopping time with respect to the independent and identically distributed interarrival times

$$\boldsymbol{Y}_{km} = \boldsymbol{X}(k2^{-m}) - \boldsymbol{X}((k-1)m), \qquad k \in \mathbb{N}$$

associated with the renewal process $\mathbf{N}_m$ we obtain by Wald's identity (Çınlar (1975)) and relations (2) and (16) that

$$r + \mathbb{E}(\boldsymbol{W}_m(r)) = \mathbb{E}(\boldsymbol{Y}_{1m})\mathbb{E}(\boldsymbol{N}_m(r) + 1) = 2^{-m}\mu_1 \mathbb{E}(\boldsymbol{N}_m(r) + 1). \tag{32}$$

Applying now relation (15) yields the desired result.□

We next mention another well-known result in renewal theory also related to the key renewal theorem.

Lemma 2.6. If $\boldsymbol{X}$ is an increasing Lévy process on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the cdf F_1 is non-arithmetic and has a positive finite first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then for every $m \in \mathbb{N}$ there exist random variables $\boldsymbol{W}_m(\infty)$ and $\boldsymbol{V}_m(\infty)$ satisfying

$$\boldsymbol{W}_m(r) \xrightarrow{d} \boldsymbol{W}_m(\infty), \quad \boldsymbol{V}_m(r) \xrightarrow{d} \boldsymbol{V}_m(\infty) \quad (r \to \infty)$$

with $\xrightarrow{d}$ denoting convergence in distribution and for every $x > 0$

$$\mathbb{P}(\boldsymbol{V}_m(\infty) \leq x) = \mathbb{P}(\boldsymbol{W}_m(\infty) \leq x) = \frac{1}{2^{-m}\mu_1}\int_0^x (1 - F_{2^{-m}}(y))dy. \tag{33}$$

Proof. As shown in Lemma 2.4 the cdf $F_{2^{-m}}$ is non-arithmetic for every $m \in \mathbb{N}$. Since μ_1 is finite and positive and $\mathbb{E}(\boldsymbol{X}(2^{-m})) = 2^{-m}\mu_1$ this shows the result (see page 194 and 195 of Karlin and Taylor (1975)).□

Clearly it follows by Lemma 2.6 that for every $m \in \mathbb{N}$ and $\alpha > 0$

$$\mathbb{E}\left(e^{-\alpha \boldsymbol{V}_m(\infty)}\right) = \mathbb{E}\left(e^{-\alpha \boldsymbol{W}_m(\infty)}\right) = \frac{2^m(1 - \mathbb{E}\left(e^{-\alpha \boldsymbol{X}(2^{-m})}\right))}{\alpha \mu_1}. \tag{34}$$

We finally introduce in this section for every $m \in \mathbb{N}$ and $x > 0$ given the function $H_m: (0, \infty) \to \mathbb{R}_+$ defined by

$$H_m(\alpha) := \frac{1}{\alpha} \mathbb{P}(\boldsymbol{W}_m(\boldsymbol{R}_\alpha) > x) = \int_0^\infty e^{-\alpha r}\, \mathbb{P}(\boldsymbol{W}_m(r) > x) dr. \tag{35}$$

with the random variable $\boldsymbol{R}_\alpha$ independent of the stochastic residual life process $\boldsymbol{W}_m = \{\boldsymbol{W}_m(r): r > 0\}$ having an exponential distribution with parameter $\alpha > 0$. For this function one can show the following result and this result will be useful in the next section to determine the representation of $\mathbb{P}(\boldsymbol{W}(r) > x)$ for any given x and $r > 0$. By relation (14) and the sequential continuity property of probability measures (Çınlar (2011)) or the Lebesque dominated convergence theorem (Rudin (1982)) it follows for every $x > 0$

$$\lim_{m\uparrow\infty} \mathbb{P}(\boldsymbol{W}_m(r) > x) \downarrow \mathbb{P}(\boldsymbol{W}(r) > x) \tag{36}$$

and for every $\alpha > 0, x > 0$

$$\lim_{m\uparrow\infty} H_m(\alpha) \downarrow H_\infty(\alpha) \tag{37}$$

with the function $H_\infty: (0, \infty) \to \mathbb{R}_+$ given by

$$H_\infty(\alpha) = \int_0^\infty e^{-\alpha r}\, \mathbb{P}(\boldsymbol{W}(r) > x) dr \tag{38}$$

Lemma 2.7. If $\boldsymbol{X}$ is an increasing Lévy process on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, then for every $\alpha > 0$ and $m \in \mathbb{N}$

$$H_m(\alpha) = \frac{\int_0^\infty e^{-\alpha r}(1 - F_{2^{-m}}(r + x))\, dr}{1 - \mathbb{E}(e^{-\alpha \boldsymbol{X}(2^{-m})})} \tag{39}$$

Proof. Since the random variable $\boldsymbol{W}_m(r)$ represents the residual life at time r of the renewal process $\boldsymbol{N}_m$ we obtain, conditioning on $\boldsymbol{X}(2^{-m})$ and applying the renewal argument, that the function $r \to \mathbb{P}(\boldsymbol{W}_m(r) > x)$ satisfies the renewal type equation

$$\mathbb{P}(\boldsymbol{W}_m(r) > x) = 1 - F_{2^{-m}}(r + x) + \int_0^r (\mathbb{P}(\boldsymbol{W}_m(r - t) > x) \mathrm{d}F_{2^{-m}}(t)$$

This shows due the convolution structure of this equation and applying Fubini's theorem (Çınlar (2011)) to the second part of the above integral equation that

$$H_m(\alpha) = \int_0^\infty e^{-\alpha r}(1 - F_{2^{-m}}(r + x))\, dr + H_m(\alpha)\left(1 - \mathbb{E}\left(e^{-\alpha \boldsymbol{X}(2^{-m})}\right)\right).$$

This shows the desired representation.□

Using Lemma 2.2 and the results for renewal processes in Lemma up 2.4 up to 2.7 we present in the next section in Theorem 3.1 the main results for the expected hitting time $\mathbb{E}(\boldsymbol{T}(r))$ and in Theorem 3.2 the main results for the expected overshoot $\mathbb{E}(\boldsymbol{W}(r))$. Moreover, in Theorem 3.5 up to Theorem 3.10 we present the main known results about the cdf of the overshoot and undershoot stochastic process by giving an alternative proof of these results using the previous mentioned result in renewal theory avoiding the compensation formula for predictable processes.

On the derivation of the distribution of the overshoot and undershoot stochastic process for increasing Lévy processes using results from renewal theory.

In this section we will use Lemma 2.2 and the results about renewal processes shown in the previous section to derive corresponding results for increasing Lévy processes. The result in relation (40) is also shown in Chapter 5 of Kyprianou (2006) using a different proof technique. The results in relations (41) and (42) seem to be new. Observe the result in relation (42) can be seen as a Lorden-type inequality for the expected hitting time $\mathbb{E}(\boldsymbol{T}(r))$ of level r.

Theorem 3.1. If $\boldsymbol{X}$ is an increasing Lévy process on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the cdf F_1 is non-arithmetic and has a finite positive first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then for every $h > 0$

$$\lim_{r\uparrow\infty}\mathbb{E}\big(\boldsymbol{T}(r+h)-\boldsymbol{T}(r)\big) = \frac{h}{\mu_1}. \tag{40}$$

If the second moment $\mu_2 = \mathbb{E}\big(\boldsymbol{X}^2(1)\big)$ is finite, then

$$\lim_{r\uparrow\infty}\mathbb{E}\big(\boldsymbol{T}(r)\big) - \frac{r}{\mu_1} = \frac{\sigma^2\big(\boldsymbol{X}(1)\big)}{2\mu_1^2} \tag{41}$$

and for every $r > 0$

$$0 \leq \mathbb{E}\big(\boldsymbol{T}(r)\big) - \frac{r}{\mu_1} \leq \frac{\sigma^2\big(\boldsymbol{X}(1)\big)}{\mu_1^2}. \tag{42}$$

Proof. It follows by relation (8) and introducing the random variable $\boldsymbol{D}_m(r,h) := \boldsymbol{T}_m(r+h) - \boldsymbol{T}_m(r)$ and $\boldsymbol{D}_\infty(r,h) := \boldsymbol{T}(r+h) - \boldsymbol{T}(r)$ that for every $m \in \mathbb{N}$ we obtain

$$\boldsymbol{D}_m(r,h) - 2^{-m} \leq \boldsymbol{D}_\infty(r,h) \leq \boldsymbol{D}_m(r,h) + 2^{-m}. \tag{43}$$

Applying Lemma 2.4 to the above inequality implies for every $m \in \mathbb{N}$ that

$$\frac{h}{\mu_1} - 2^{-m} \leq \lim\ \inf_{r\uparrow\infty}\mathbb{E}\big(\boldsymbol{D}_\infty(r,h)\big) \leq \lim\sup_{r\uparrow\infty}\ \mathbb{E}\big(\boldsymbol{D}_\infty(r,h)\big) \leq \frac{h}{\mu_1} + 2^{-m} \tag{44}$$

Since $m \in \mathbb{N}$ is arbitrary in relation (44) the result in relation (40) follows. To show relation (41) we observe again by relation (8) that for every $r > 0$ and $m \in \mathbb{N}$

$$\mathbb{E}\big(\boldsymbol{T}_m(r)\big) - \frac{h}{\mu_1} - 2^{-m} \leq \mathbb{E}\big(\boldsymbol{T}(r)\big) - \frac{h}{\mu_1} \leq \mathbb{E}\big(\boldsymbol{T}_m(r)\big) - \frac{h}{\mu_1}. \tag{45}$$

Applying relation (27) to the upper and lower bound in relation (45) and $m \in \mathbb{N}$ is arbitrary we obtain the result given in relation (41). To show relation (42) we observe by the monotone convergence theorem (Çınlar (2011), Rudin (1982)) and relation (9) that $\lim_{m\uparrow\infty}\mathbb{E}\big(\boldsymbol{T}_m(r)\big) \downarrow \mathbb{E}\big(\boldsymbol{T}(r)\big)$. Applying relation (28) this shows the result in relation (42).□

We now mention the following result for the expectation of the overshoot of an increasing Lévy process. This result in relation (47) seems to be new.

Theorem 3.2. If $\boldsymbol{X}$ is an increasing Lévy process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the cdf $F_1$is non-arithmetic and has a finite second moment $\mu_2 = \mathbb{E}\left(\boldsymbol{X}^2(1)\right)$, then

$$\lim_{r\uparrow\infty}\mathbb{E}\big(\boldsymbol{W}(r)\big) \leq \frac{\sigma^2\big(\boldsymbol{X}(1)\big)}{2\mu_1} \tag{46}$$

and for every $r > 0$

$$0 \leq \mathbb{E}\big(\boldsymbol{W}(r)\big) \leq \frac{\sigma^2\big(\boldsymbol{X}(1)\big)}{\mu_1} \tag{47}$$

Proof. Applying relations (9) and (14) and the monotone convergence theorem we obtain using Lemma 2.5 that $\mathbb{E}\big(\boldsymbol{W}(r)\big) = \mu_1\mathbb{E}\big(\boldsymbol{T}(r)\big) - r$ for every $r > 0$. By Theorem 3.1 this shows the result.□

We denote the Laplace-Stieltjes transform (LST) of $\boldsymbol{X}(t)$ by π_t, that is

$$\pi_t(\alpha) := \mathbb{E}\big(e^{-\alpha \boldsymbol{X}(t)}\big). \tag{48}$$

Since in an increasing Lévy process the cdf of $\boldsymbol{X}(t)$ is concentrated on $(0,\infty)$ and is infinitely divisible it is well-known (see Theorem 4.2 and 4.3 in Chapter 3 of Steutel and van Harn (2004)) that the next representation holds for $\mathbb{E}\big(e^{-\alpha\boldsymbol{X}(t)}\big)$ for every $\alpha > 0$ and $t > 0$.

Lemma 3.3. It follows for every $\alpha > 0$ and $t > 0$ that $\pi_t(\alpha) = \pi_1(\alpha)^t$ with

$$\pi_1(\alpha) = e^{-\int_0^\alpha \rho(s)ds}, \rho(s) = \int_{0^-}^{\infty} e^{-sx}dK(x). \tag{49}$$

and $K: \mathbb{R} \mapsto \mathbb{R}$ a (right continuous) nondecreasing function satisfying $K(x) = 0$ for every $x < 0$.

By Lemma 3.3 it follows for every $\alpha > 0$ and $t > 0$ that $\pi'_t(\alpha) = -\rho(\alpha)\pi_t(\alpha)$ with π'_t denoting the derivative of the function π_t. Since $\mu_1 := \mathbb{E}(\boldsymbol{X}(1))$is finite and positive and using

$$\mu_1 = -\lim_{\alpha\downarrow 0}\pi'_1(\alpha) := -\pi'_1(0^+)$$

we obtain that $\mu_1 = \rho(0^+) = K(\infty)$. Hence the normalized right continuous function $\bar{K}$: $[0,\infty) \to [0,1]$ given by

$$\bar{K}(x) := \frac{K(x)}{\mu_1} \tag{50}$$

can be seen as the cdf of a nonnegative random variable $\boldsymbol{Z}$. The next result plays a crucial role in showing the correctness of our approximation technique by means of renewal processes.

Lemma 3.4. If $\boldsymbol{X}$ is an increasing Lévy process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and a finite positive first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then for every $x_1, x_2 > 0$

$$\lim_{t\downarrow 0}\frac{F_t(x_1 + x_2) - F_t(x_1)}{t} := \int_{x_1}^{x_1+x_2} u^{-1}dK(u) \tag{51}$$

with K a cdf on $\mathbb{R}_+$. Moreover,

$$\lim_{t\downarrow 0}\frac{1-F_t(x_1)}{t} := \int_{x_1}^{\infty} u^{-1}dK(u). \tag{52}$$

Proof. By Lemma 3.3 and relation (50) we obtain

$$-\int_{0^-}^{\infty} x\exp(-\alpha x)\,dF_t(x) = \pi'_t(\alpha) = -t\int_{0^-}^{\infty}\exp(-\alpha x)dK(x)\pi_t(\alpha) \tag{53}$$

with $K = \mu_1\bar{K}$ and $\bar{K}$ a cdf on $\mathbb{R}_+$. Introducing for every $t > 0$ the function $L_t\colon[0,\infty)\to[0,\infty)$ given by

$$L_t(x) := t^{-1}\int_0^x u dF_t(u) \tag{54}$$

it follows by relation (53) and Laplace inversion that

$$L_t(x) := \mu_1\int_0^x F_t(x-u)d\,\bar{K}(u) \tag{55}$$

for every $t > 0$. By relation (55) and the interpretation after relation (50) the value $L_t(x)$ can be seen as

$$L_t(x) = \mu_1\mathbb{P}(\boldsymbol{X}(t)+\boldsymbol{Z}\le x) \tag{56}$$

with the random variable $\boldsymbol{X}(t)$ independent of the nonnegative random variable $\boldsymbol{Z}$ and

$$\mathbb{P}(\boldsymbol{Z}\le x) = \bar{K}(x)$$

This shows that the function $x\mapsto\frac{L_t(x)}{\mu_1}$ is a cdf on $\mathbb{R}_+$ for every $t > 0$. Since the stochastic process $\boldsymbol{X}$ has increasing sample paths it also follows using relation (56) that the function $t\mapsto L_t(x)$ is decreasing for every fixed x. Finally, by the continuity in probability and $\boldsymbol{X}(0) = 0$ we obtain applying again relation (56) that

$$\lim_{t\downarrow 0}L_t(x) = \mu_1\mathbb{P}(\boldsymbol{Z}\le x) = K(x). \tag{57}$$

Hence this shows (as we already know) that the function $x\mapsto\frac{K(x)}{\mu_1}$ is a cdf and so

$$\int_x^{\infty} u^{-1}d\,K(u) < \infty$$

for every $x > 0$. Since for every $x_1, x_2 > 0$ it follows by relation (54) that

$$\frac{F_t(x_1+x_2)-F_t(x_1)}{t} = \int_{x_1}^{x_1+x_2} u^{-1}dL_t(u) \tag{58}$$

we obtain by relation (57) and the Helly-Bray lemma (Loeve (1977)) that

$$\lim_{t\downarrow 0}\frac{F_t(x_1+x_2)-F_t(x_1)}{t} = \lim_{t\downarrow 0}\int_{x_1}^{x_1+x_2} u^{-1}dL_t(u) = \int_{x_1}^{x_1+x_2} u^{-1}dK(u).$$

Applying the extended Helly-Bray Lemma (Loeve (1977)) we also obtain that

$$\lim_{t\downarrow 0}\frac{1-F_t(x_1)}{t}=\int_{x_1}^{\infty}u^{-1}dK(u)$$

and this shows the result.□

In the next result we will give for every $r > 0$ a more detailed expression of the function $H_\infty : (0,\infty) \to \mathbb{R}_+$ listed in relation (38) for every $x > 0$ using a limiting argument applying relation (37), Lemma 3.4 and Lemma 2.7. Since the function H_∞ is related to the Laplace-Stieltjes transform of the function $r \to \mathbb{P}(\boldsymbol{W}(r) > x)$ for any $x > 0$ we can use this representation to give an expression for $\mathbb{P}(\boldsymbol{W}(r) > x)$, thereby recovering a known result obtained by the compensation formula for predictable processes (Kyprianou (2006), Bertoin et al. (1999)).

Theorem 3.5. If $\boldsymbol{X}$ is an increasing Lévy process on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and a finite positive first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then for every $\alpha > 0$ and $x > 0$

$$H_\infty(\alpha)=\frac{\int_0^\infty e^{-\alpha r}\int_{r+x}^\infty u^{-1}dK(u)\,dr}{\int_0^\alpha \rho(s)ds} \tag{59}$$

with H_∞ listed in relation (38). Also, it follows for every $x > 0$ and $r > 0$ that

$$\mathbb{P}(\boldsymbol{W}(r)>x)=\int_0^r\int_{r+x-y}^\infty u^{-1}dK(u)\,dG_\infty(y). \tag{60}$$

with $G_\infty(y) := \mathbb{E}(\boldsymbol{T}(r)), r \geq 0$.

Proof. Introduce for every m∈ N the functions $G_m\colon (0,\infty)\mapsto \mathbb{R}_+$ given by

$$G_m(r) := \mathbb{E}(\boldsymbol{T}_m(r)). \tag{61}$$

It follows by relation (9) and the monotone convergence theorem that

$$\lim_{m\uparrow\infty} G_m(r) = G_\infty(r) \tag{62}$$

for every $r > 0$. Also, by relation (15) we obtain

$$G_m(r) = 2^{-m}\mathbb{E}(\boldsymbol{N}_m(r)+1) = 2^{-m}U_m(r)$$

with $U_m(r) := \sum_{k=0}^{\infty} F_{2^{-m}}^{k_*}(r)$ the well-known renewal function of the renewal process $\boldsymbol{N}_m$. Since for every $\alpha > 0$

$$\frac{2^{-m}}{1-\mathbb{E}(e^{-\alpha\boldsymbol{X}(2^{-m})})}=2^{-m}\int_{0^-}^\infty e^{-\alpha r}\,dU_m(r)=\int_{0^-}^\infty e^{-\alpha r}\,dG_m(r). \tag{63}$$

this implies by the extended Helly-Bray theorem (Loeve (1977)) using Lemma 3.3 and relation (62) that

$$\left(\int_0^\alpha \rho(s)ds\right)^{-1}=\lim_{m\uparrow\infty}\frac{2^{-m}}{1-e^{-2^{-m}\int_0^\alpha\rho(s)ds}}=\lim_{m\uparrow\infty}\frac{2^{-m}}{1-\mathbb{E}(e^{-\alpha\boldsymbol{X}(2^{-m})})}=\int_0^\infty e^{-\alpha r}dG_\infty(r). \tag{64}$$

Also, by Markov's inequality (see Section 2.8 of Ross (2010)) and relation (2) we obtain for every $m \in \mathbb{N}$ and $r > 0, x > 0$ that

$$2^m\big(1 - F_{2^{-m}}(r + x)\big) \le \frac{2^m \mathbb{E}\big(\boldsymbol{X}(2^{-m})\big)}{r + x} = \frac{\mu_1}{r + x} \tag{65}$$

Hence by the Lebesque dominated convergence theorem we obtain for any $x > 0$ using relation (52) that

$$\lim_{m\uparrow\infty} 2^m \int_0^\infty e^{-\alpha r}\big(1 - F_{2^{-m}}(r + x)\big)dr = \int_0^\infty e^{-\alpha r}\int_{r+x}^\infty u^{-1}dK(u)\,dr. \tag{66}$$

Applying relations (37), (64), (66) and Lemma 2.7 we obtain relation (59). Since it is easy to check using relation (59) and (64) that for every $\alpha > 0$

$$\int_0^\infty e^{-\alpha r}\int_0^r \int_{r+x-y}^\infty u^{-1}dK(u)\,dG_\infty(y)\,dr = \left(\int_0^\infty e^{-\alpha r}\,dG_\infty(y)\right)\left(\int_0^\infty e^{-\alpha r}\int_{r+x}^\infty u^{-1}dK(u)\,dr\right) = H_\infty(\alpha) \tag{67}$$

the result in relation (60) follows due to the one-to-one relation between a Laplace transform and its underlying function.□

To give an alternative interpretation of $G_\infty(r)$ related to the so-called α-potential, $\alpha \ge 0$ (Kyprianou (2006)) given by

$$U^{(\alpha)}(x) := \int_0^\infty e^{-\alpha t}\mathbb{P}(\boldsymbol{X}(t) \le x)dt \tag{68}$$

we observe that

$$G_\infty(r) = \mathbb{E}\big(\boldsymbol{T}(r)\big) = \int_0^\infty \mathbb{P}(\boldsymbol{T}(r) > t)dt = \int_0^\infty \mathbb{P}(\boldsymbol{X}(t) \le r)dt = U^{(0)}(r) \tag{69}$$

Applying relation (14) in combination with either relation (24) or relation (25) it is easy to determine the cdf of the random variable $\boldsymbol{V}(r)$ or the random vector $(\boldsymbol{V}(r), \boldsymbol{W}(r))$ for r fixed. Applying these relations, we only need to know as shown in the next proof the cdf of the random variable $\boldsymbol{W}(r)$ presented in Theorem 3.5. Since clearly $\boldsymbol{V}(r) \le r$ it follows for every $x \ge r > 0$ that

$$\mathbb{P}(\boldsymbol{V}(r) > x) = 0.$$

In the next lemma result we recover a known result (Kyprianou (2006),Bertoin et al. (1999)).

Corollary 3.6. If $\boldsymbol{X}$ is an increasing Lévy process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ and a finite positive first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then for every $r > x > 0$

$$\mathbb{P}(\boldsymbol{V}(r) > x) = \int_0^{r-x}\int_{r-y}^\infty u^{-1}dK(u)\,dG_\infty(y). \tag{70}$$

Proof. It follows by the Lebesgue dominated convergence theorem (rewrite $\mathbb{P}(\boldsymbol{Z} > x) = \mathbb{E}(1_{\{\boldsymbol{Z}>x\}}!)$ for any random variable $\boldsymbol{Z}$) and relations (14) and (18) that for every $r > x > 0$

$$\lim_{m\uparrow\infty}\mathbb{P}(\boldsymbol{V}_m(r) > x) = \mathbb{P}(\boldsymbol{V}(r) > x) \tag{71}$$

and

$$\lim_{m\uparrow\infty}\mathbb{P}(\boldsymbol{W}_m(r) > x) = \mathbb{P}(\boldsymbol{W}(r) > x) \tag{72}$$

This implies using relation (24) that for every $r > x$

$$\mathbb{P}(\boldsymbol{V}(r) > x) = \mathbb{P}(\boldsymbol{W}(r-x) > x) \tag{73}$$

The desired result follows by Theorem 3.5.□

Since by Lebesque's dominated convergence theorem and relations (14) and (18) it also follows

$$\lim_{m\uparrow\infty}\mathbb{P}(\boldsymbol{V}_m(r) > x, \boldsymbol{W}_m(r) > w) = \mathbb{P}(\boldsymbol{V}(r) > x, \boldsymbol{W}(r) > w) \tag{74}$$

we can apply a similar proof as in Theorem 3.6 using relation (25) to show the following known result for the joint cdf of the random vector $(\boldsymbol{V}(r), \boldsymbol{W}(r))$ (Kyprianou (2006), Bertoin et al. (1999)).

Corollary 3.7. If $\boldsymbol{X}$ is an increasing Lévy process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and a finite positive first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then for every $r > x > 0$ and $w > 0$

$$\mathbb{P}(\boldsymbol{V}(r) > x, \boldsymbol{W}(r) > w) = \int_0^{r-x}\int_{r+w-y}^{\infty} u^{-1} dK(u)\, dG_\infty(y). \tag{75}$$

Finally, we show the following result for the limiting cdf of the random variable W(r) by applying Theorem 3.5 and copying the proof of Theorem 3.8 at page 295 of Çınlar (1975). This proof is in line with standard proofs in renewal theory. An alternative proof of this result will also be listed using the results in Lemma 2.2 and Lemma 3.3. The second proof deals more explicitly with changing the limit operations as mentioned in the quote given in Bertoin et al. (1999).

Theorem 3.8. If $\boldsymbol{X}$ is an increasing Lévy process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the cdf F_1 is non-arithmetic and has a positive finite first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then there exists a random variable $\boldsymbol{W}(\infty)$ satisfying

$$\boldsymbol{W}(r) \overset{d}{\to} \boldsymbol{W}(\infty)\ (r \to \infty), \tag{76}$$

and this random variable $\boldsymbol{W}(\infty)$ satisfies

$$\mathbb{P}(\boldsymbol{W}(\infty) \leq x) = \frac{1}{\mu_1}\int_0^x\int_z^\infty u^{-1} dK(u)\, dz.$$

Proof. (Version 1). It follows by Theorem 3.5 that

$$\mathbb{P}(\boldsymbol{W}(r) > x) = \frac{1}{\mu_1}\int_0^x g(r-z) dG_\infty(z). \tag{77}$$

with $g: (0,\infty) \to \mathbb{R}_+$ given by

$$g(z) := \int_{z+x}^\infty u^{-1} dK(u). \tag{78}$$

Since by Lemma 3.3 and relation (50) the function $K = \mu_1 \bar{K}$ and $\bar{K}$ a cdf we obtain that the function g is nonincreasing, nonpositive and finite. Also, it follows by Fubini's theorem that for every $x > 0$

$$\int_0^\infty g(z)dz = \int_x^\infty \int_z^\infty u^{-1} dK(u)\, dy = \int_x^\infty \int_x^u dz u^{-1} dK(u) = \int_x^\infty \frac{u-x}{u} dK(u) \le K(\infty) = \mu_1 \quad (79)$$

Hence by Proposition 2.16 at page 296 of Çınlar (1975) it follows that the function g is directly Riemann integrable. Applying relation (77) and Theorem 2.8 on page 295 of Çınlar (1975) we obtain that

$$\lim_{r\uparrow\infty} \mathbb{P}(\boldsymbol{W}(r) > x) = \lim_{r\uparrow\infty} \int_0^r g(r-z) dG_\infty(z). = \frac{1}{\mu_1} \int_0^\infty g(z)dz. \quad (80)$$

Since it is easy to verify using a similar approach as in relation (79) (take $x = 0$!) that

$$\int_0^\infty \int_z^\infty u^{-1} dK(u)\, dz = \mu_1$$

this implies the result using relations (79) and (80).□

Proof. (Version 2) Introducing the LST

$$\pi_{\boldsymbol{W}_m(r)}(\alpha) := \mathbb{E}\left(e^{-\alpha \boldsymbol{W}_m(r)}\right)$$

we obtain by Lemma 2.2 for every $\alpha \ge 0$, $m \in \mathbb{N}$ and $r > 0$ that

$$\pi_{\boldsymbol{W}_m(r)}(\alpha) \le \mathbb{E}\left(e^{-\alpha \boldsymbol{W}(r)}\right) \le \frac{\pi_{\boldsymbol{W}_m(r)}(\alpha)}{\mathbb{E}\left(e^{-\alpha \boldsymbol{X}(2^{-m})}\right)}. \quad (81)$$

This implies by Lemma 3.3 that for any $m \in \mathbb{N}$, $r > 0$

$$\pi_{\boldsymbol{W}_m(r)}(\alpha) \le \mathbb{E}\left(e^{-\alpha \boldsymbol{W}(r)}\right) \le \pi_{\boldsymbol{W}_m(r)}(\alpha) \exp\left(2^{-m} \int_0^\alpha \rho(s) ds\right). \quad (82)$$

In the remainder of the proof, we will show that $\lim_{r\uparrow\infty} \mathbb{E}\left(e^{-\alpha \boldsymbol{W}(r)}\right)$ exists and identify its limit. Applying Lemma 2.6 and the continuity theorem for LST (Feller (1971) or Appendix A3 of Steutel and van Harn (2004)) and Lemma 3.3 and relation (34) we obtain for every $m \in \mathbb{N}$

$$\lim_{r\uparrow\infty} \pi_{\boldsymbol{W}_m(r)}(\alpha) = \mathbb{E}(\exp(-\alpha \boldsymbol{W}_m(\infty))) = \frac{2^m \left(1 - e^{-2^{-m} \int_0^\alpha \rho(s) ds}\right)}{\alpha \mu_1}. \quad (83)$$

Introduce now for every $\alpha \ge 0$ the functions

$$\bar{L}(\alpha) := \lim\, sup_{r\uparrow\infty} \mathbb{E}(\, exp(-\alpha \boldsymbol{W}(r))$$

and

$$\underline{L}(\alpha) := \lim\, inf_{r\uparrow\infty} \mathbb{E}(\, exp(-\alpha \boldsymbol{W}(r))$$

By relations (82) and (83) we obtain for every $m \in \mathbb{N} > 0$ that

$$\mathbb{E}(\exp(-\alpha \boldsymbol{W}_m(\infty))) \le \underline{L}(\alpha) \le \bar{L}(\alpha) \le \mathbb{E}\left(\exp\left(-\alpha \boldsymbol{W}_m(\infty)\right)\right) \exp\, (2^{-m} \int_0^\alpha \rho(s) ds). \quad (84)$$

By taking $m \uparrow \infty$ in relation (84) and using relation (83) we find

$$\frac{1}{\alpha\mu_1}\int_0^\alpha \rho(s)ds \leq \underline{L}(\alpha) \leq \bar{L}(\alpha) \leq \frac{1}{\alpha\mu_1}\int_0^\alpha \rho(s)ds. \tag{85}$$

Hence $lim_{r\uparrow\infty}\mathbb{E}(e^{-\alpha \boldsymbol{W}(r)})$ exists for every $\alpha \geq 0$ and

$$lim_{r\uparrow\infty}\mathbb{E}(e^{-\alpha \boldsymbol{W}(r)}) = \frac{1}{\alpha\mu_1}\int_0^\alpha \rho(s)ds. \tag{86}$$

Using $K = \mu_1 \bar{K}$ and $\bar{K}$ a cdf on $\mathbb{R}_+$ (see Lemma 3.3), we obtain that the function ρ is continuous on $(0,\infty)$ and right continuous in 0 satisfying $\rho(0) = \mu_1$. This implies

$$\lim_{\alpha\downarrow 0}\alpha^{-1}\int_0^\alpha \rho(s)ds = \rho(0) = \mu_1. \tag{87}$$

and we have verified that the function $\alpha \mapsto \alpha^{-1}\int_0^\alpha \rho(s)ds$ is right continuous in 0. To identify the cdf with LST $\alpha \mapsto \alpha^{-1}\int_0^\alpha \rho(s)ds$ we observe by Fubini's theorem

$$\begin{aligned}\alpha^{-1}\int_0^\alpha \rho(s)ds &= \alpha^{-1}\int_0^\alpha\int_0^\infty \exp(-sx)\,dK(x)ds = \alpha^{-1}\int_0^\infty \frac{1-\exp(-\alpha x)}{x}dK(x)\\ &= \int_0^\infty\int_0^x \exp(-\alpha v)\,dv x^{-1}dK(x) = \int_0^\infty \exp(-\alpha v)\int_v^\infty x^{-1}dK(x)\,dv.\end{aligned}$$

Hence by the one-to one relation between a Laplace-Stieltjes transform and its underlying function and the continuity theorem for LST (Steutel and van Harn (2004), Feller (1971) and relation (86) the desired result follows.□

Next, we show a similar result for the limit distribution of the undershoot stochastic process making use of relation (24) and applying a limiting argument. Again, this result is known in the literature using the compensation formulas for predictable processes (Kyprianou (2006)).

Corollary 3.9. If $\boldsymbol{X}$ is an increasing Lévy process on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the cdf F_1 is non-arithmetic and has a positive finite first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then there exists some random variable $\boldsymbol{V}(\infty)$ satisfying

$$\boldsymbol{V}(r) \overset{d}{\rightarrow} \boldsymbol{V}(\infty)\ (r \uparrow \infty) \tag{88}$$

and this random variable $\boldsymbol{V}(\infty)$ satisfies

$$\mathbb{P}(\boldsymbol{V}(\infty) \leq x) = \frac{1}{\mu_1}\int_0^x\int_z^\infty u^{-1}dK(u)\,dz. \tag{89}$$

Proof. Apply relation (73) and Theorem 3.8.□

Finally, we show the following known result (Bertoin et al. (1999), Kyprianou (2006)) for the joint cdf of the random vector (V(∞),W(∞)) using relation (25) and a limiting argument.

Corollary 3.10. If $\boldsymbol{X}$ is an increasing Lévy process on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ having filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the cdf F_1 is non-arithmetic and has a positive finite first moment $\mu_1 = \mathbb{E}(\boldsymbol{X}(1))$, then there exists some random variable $(\boldsymbol{V}(\infty),\boldsymbol{W}(\infty))$ satisfying

$$(\boldsymbol{V}(r),\boldsymbol{W}(r)) \overset{d}{\rightarrow} (\boldsymbol{V}(\infty),\boldsymbol{W}(\infty))\ (r \uparrow \infty) \tag{90}$$

and this random vector $(\boldsymbol{V}(\infty), \boldsymbol{W}(\infty))$ satisfies

$$\mathbb{P}(\boldsymbol{V}(\infty) > x, \boldsymbol{W}(\infty) > w) = \frac{1}{\mu_1} \int_{x+w}^{\infty} \int_{z}^{\infty} u^{-1} dK(u)\, dz. \tag{91}$$

Proof. By relations (72) and (25) it follows for every $r > x$ and $w > 0$ that

$$\mathbb{P}(\boldsymbol{W}(r-x) > x+w) = \mathbb{P}(\boldsymbol{V}(r) > x, \boldsymbol{W}(r) > w). \tag{92}$$

The result now follows by applying Theorem 3.8.□

Summary and directions of future research

The sole and only purpose of this paper is to provide an elementary way to compute the cdf and limiting cdf of the overshoot and undershoot stochastic process for an increasing Lévy process by relating this Lévy process to a renewal process. This relation with renewal theory is already mentioned in the literature Bertoin et al. (1999) but to verify these limiting results the compensation formula for predictable processes is used (Bertoin et al. (1999)). Instead of using this formula this paper uses explicitly the relation between renewal processes and increasing Lévy processes. In particular, proofs of known results for hitting times and the overshoot and undershoot stochastic process in an increasing Lévy process are given by making only use of well-known results from renewal theory and the one-to-one relation between Laplace-Stieltjes transforms and the underlying function. As such this more elementary approach might be beneficial to more practical oriented researchers within maintenance and risk theory familiar with renewal theory and not so familiar with the more advanced theory of predictable processes.

Since in the literature convergence rate results of the renewal measure to the Lebesque measure are known (these results can be shown by either coupling methods (Hermann (2000)) or Banach algebra techniques (Frenk (1987))) it would be natural to investigate how fast the cdf of the random variable $\boldsymbol{W}(r)$ or $\boldsymbol{V}(r)$ will convergence to the limiting cdf. These convergence rates depend on the behavior of the tail of the cdf $\bar{K}$ given in relation (50). Although this might seem an academic problem convergence rates for the renewal measure immediately imply convergence rate results of, for example, Markov processes to the equilibrium state. This means in a lot of real-life applications modelled by Markov processes or in general regenerative stochastic processes one obtains an indication of the speed of convergence to the equilibrium state. In our particular case, these convergence rate results for the cdf of $\boldsymbol{W}(r)$ give an indication for which values of r we may use the limiting cdf as a good approximation of the original cdf. Since it seems difficult at first sight to apply coupling methods to Lévy processes another way to achieve rate of convergence results is to apply Banach algebra techniques to the LST of the random variable $\boldsymbol{W}(r)$. Investigating this possibility can be a future line of research. Banach algebra techniques applied to the Laplace transform of the renewal function seem to achieve according to the authors knowledge the best possible convergence rate results. However, although an active field of research in the past, these two different techniques are not often applied and improved anymore and so most of the convergence rate results for the renewal measure to the Lebesque measure achieved by these two different methods are more than 20 years old.

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