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RESEARCH PAPER

Gift Card Pricing through its Pre-sales and Ordering Decisions in a Two-echelon Supply Chain with a Separate Sales Channel and Price-sensitive Demand

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Abstract

The high popularity and profitability of gift cards encourage many sellers to use them to sell their goods. Retailers have also been encouraged to use independent third parties to sell their gift cards for increasing their sales channel and taking advantage of it. This paper develops a two-echelon supply chain for gift card incentive policy, with a third party and retailer at the first level and a supplier at the second one. The most important research questions are as follows: order amount of chain members to maximize their own and the whole chain profit, gift card prices by the retailer to its customers, gift card prices by the retailer to third party, and gift card prices by the third party to customers. Stackelberg's approach is used to solve the model, assuming that the third party is the follower and the retailer is the leader. In addition, by proving the concavity of the objective function, obtaining the closed-form solution for variables, and proving the resulting solutions, an algorithm has been developed to achieve the optimal answer. Findings showed that the use of cards in the case of economic order models increases the demand for retail and on the other hand attracts more customers and better brand expansion. A numerical example as well as a sensitivity analysis are performed to describe the model. Finally, conclusions as well as suggestions for future research are provided.

Keywords:
Gift Card;
Supply Chain;
Incentive Policy;
EOQ Model;
Third Party.

Introduction and Literature Review

Today, sellers use different incentive mechanisms such as gift cards to maintain and expand their sales market. Gift cards affect the costs and benefits of those who offer them. Nowadays, gift cards are used around the world as a modern alternative to all kinds of gifts. Gift cards are very popular among people. The use of these cards is growing rapidly, to the point that in 2007 they created a turnover of up to $ 100 billion for companies. So it is no surprise that retailers are using gift cards as a popular way to attract customers. Mooncake gift cards, which many companies in China give to their employees, are a type of gift card. These gift cards are used for shopping during the Mooncake festival. Indeed, gift cards are a kind of discount model that is given to customers at a certain time. In terms of payment time, gift cards are divided into two categories: Free and Pre-paid gift cards. The second type of gift card not only encourages people to buy but also its buyers can give them as gifts to others.

On the other hand, gift cards can also be considered in three sections: for retailer level, for the product, and for network brand. In fact, gift cards either specifically belong to a particular retailer, to a specific product, or to a specific brand. Product-specific cards are designed In order

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to buy one or more specific goods. The names and numbers of these products are recorded on the cards. Retailer-specific cards, as their name implies, are designed to buy the goods of a particular retailer. Retailer-specific cards are used by many world-renowned retailers, including Wal-Mart, Tesco, and Carrefour Network brand cards such as American Express Gift Card [1] and Okcard issued by Bailian Group in China can be traded with any credit card. Product cards can be used to buy seasonal products. Indeed, using these gift cards will increase the demand for some seasonal products for a certain period of time. In this study, product-specific cards and prepaid gift cards will be examined.

Pre-paid product-specific gift cards have three advantages. The first advantage of these gift cards is that they increase product demand [2]. Given that these gift cards can be given as gifts to others, even customers who do not intend to buy the product or do not know the product will be encouraged to buy the product, which in turn will increase demand. The second advantage is financing the retailer in times of lack of liquidity without using high-interest loans. Since the retailer first sells gift cards to its customers and then starts selling the goods after a certain period of time, he or she can use the money earned to finance the company. This way, retailers will no longer have to use bank loans, which are often high-interest and cannot be obtained quickly when needed. The third advantage of these cards is that if they are not used by their owners, they will expire after a while. In fact, we receive card money from customers without selling any goods. Given that the price of gift cards is calculated on the assumption that these cards are sold before goods are delivered, inflation and its impact must be considered.

The following questions should be answered when using a gift card in a supply chain: Can prepaid gift cards increase supply chain profits? How much is the profit margin for each member of the chain? At what price should gift cards be sold? How much does the order of chain members change due to the gift card? In the following, articles related to gift cards and other incentive policies will be reviewed.

In order to analyze gift card performance, Khouja et al. [3] used model hypotheses developed by Cachon and Swinney [4] to analyze gift card performance. Their goal was to get the optimal value of the gift card and the level of retail inventory. They created a free gift card and assumed that each part would be divided into two parts, the product priced at "P" in the first part and the remaining stock discounted and sold in the second part. They also categorized customers into three groups. 1- Transaction hunters 2- Short vision customers, and 3- Strategic customers. Finally, they performed a discount strategy compared to the free gift card strategy. Cao et al. [5] hypothesized a company in which a new product is sold to new customers as well as commercial services to alternative consumers. Theoretical models have also been developed to examine optimal decisions about gift cards and cash and to evaluate the optimal payment for development discounts by them. Zhang et al. [6] combined the newsvendor model with pre-paid product-specific gift cards and then compared it with the classic newsvendor model. Model profit optimization has been investigated in the following three modes in this study: surplus demand generated by gift cards, cash receipts at the time of gift card sales, and gift cards without redemption. Norvell and Horky [7] conducted a survey among gift card retailers at a national restaurant chain restaurant to examine how gift cards affect customer shopping behavior. Based on this information, along with operating margins, they modeled the impact of three different gift card discount scenarios on the company's revenue and profits. Despite the positive effect of all the scenarios, these cases did not lead to profitability. In this case, the profit in the best-case scenario was significantly lower than expected, and in the worst-case scenario, it was even negative. Park and Yi [8] examine the reasons for different perceptions of donors and recipients of the discounted gift value. These studies show that donors value less discounted gifts than regular gifts, while the recipients of these gifts do not value them differently. Khoja et al. [9] also combined a newsvendor model with a gift card incentive strategy. They examined the sales results of gift cards based on the optimal availability of products sold during the holidays in the pre-holiday period and how these products were priced in the post-holiday period from the perspective of a retailer.

Given that, in the proposed model, gift cards are sold through both retailers and third parties, the literature on the multi-channel supply chain is also examined.

Khoja and Zhou [10] developed a model in which a service provider sells gift cards to its customers and a retailer. Retailers buy gift cards at a lower price from the service provider and sell them to customers at a significant profit margin. Cao et al. [11] considered a firm selling a new product to new consumers and offering a trade-in service to replacement consumers. Li et al. [12] consider a supply chain including two manufacturers and one retailer. They develop several strategies for using free gift cards in this two-product supply chain. Li et al. [13] develop a decentralized two-product supply chain in which the retailer is a Stackelberg leader. They analyze three models: no gift cards, manufacturer-sponsored gift cards, and retailer-sponsored gift cards. Lashgari et al. [14] aim to examine the effects of gift cards and inflation on optimal ordering policy for regular products and to analyze the advantages of providing product-specific gift cards. Two models are proposed. In the first model, the benefit function of the retailer and supplier is considered separately and in the second model both of them are considered as a chain, and the benefit function is optimized for a limited planning horizon.

In fact, this model includes two sales channels, the first channel is the service provider and the second channel is the retailer. In this model, the retailer plays the role of a follower and the service provider plays the role of a leader. In this study, two modes were examined: when gift cards can be given as gifts to others and when they cannot be given as gifts. Some other studies also focused on dual-channel models [15,16,17,18,19]. A supply chain of the same type along with the length of the warranty period as a determining factor in consumer preferences in cooperative and non-cooperative environments was examined by Tsao and Su [20].

Research shows that the existence of a second channel leads to an increase in supply chain profits [21]. The results of marketing research also show the fact that increasing channels will be associated with increased consumer demand and consequently with increasing profits of the entire supply chain. Arpita Roy et al. designed a two-channel model for a two-echelon supply chain in which the manufacturer sells its products online and on traditional platforms (brick and mortar). Batarfi et al. [22] created a two-tier two-channel supply chain for standard and customized products and considered the effects of learning and forgetting on production processes. Ghosh et al. [23] investigated a two-tier two-channel supply chain model with emission-sensitive random demand under government restrictions on forced trade restrictions and low-carbon consumer preferences. Panda et al. [24] reviewed pricing strategy and refund policies for a high-tech product in which the goal was to reduce the unit cost of each product during the limited life cycle in a supply chain with two channels. To solve the model and get the optimal order quantity and price, they used the Stackelberg approach in which Stackelberg is the leading manufacturer. Erwin Vidodo provides a model for evaluating the impact of an alternative product on a two-channel supply chain. To coordinate offline and online channels, he has used two important variables: order quantity and product price.

A review of the research literature reveals that no study has been conducted on product-specific gift cards in the field of EOQ models and dual-channel supply chains for conventional products. Previous models are less about using gift cards and focus on single-level and single-player models.

Therefore, in this research, a two-tier supply chain with three members and two sales channels has been developed for a product-specific gift card.

The following questions will be answered in this research:

1. What is the product selling price at the retailer level to the customers, the gift card selling price by the retailer to a third party, and the gift card selling price by the third party to its customers?

2. Achieving the optimal selling price of the card to a third party with the aim of increasing the profit of the whole supply chain
3. What is the optimal order amount for the retailer and supplier? Also, what is the optimal selling price of a gift card by a retailer?
4. Is having a third party profitable for chain members?
5. Is there a feasible price for a gift card to grow the capital profit of the whole chain and create a regular order schedule for all the SC parties at the same time?
6. What are the effects of the percentage discount of gift cards on the model?
7. What are the effects of inflation on the model?
8. What are the effects of third parties on the model?
9. What are the effects of the sales price of third parties on the whole supply chain profits?

To answer the above questions, a two-tier supply chain has been designed, the first tier has two members (retailer and third party) and the second tier has one member (supplier). In the first level, the retailer uses two sales channels to sell the gift card, and this model also includes inflation.

The model presented is such that a retailer sells gift cards in two other ways, in addition to selling the goods in cash to its customers. The first way is selling gift cards directly to institutions, companies, and customers. The second way is selling gift cards to third parties, and finally to sell gift cards by third parties to other customers. In this situation, the retailer uses the EOQ model to supply the goods, and orders are made on this basis. The goal is to achieve the maximum profit of the supply chain and the profit of each member of the supply chain in the face of inflation. Here the supply chain is configured under two different conditions:

• A model in which each part of the chain makes decisions based solely on reliable profits.

• An open model in which each department is involved to maximize the capital gain of the entire system.

In the first model, each member seeks to optimize their profits, but in the second model, decisions are made collectively.

The main innovations of this research are as follows:

- Combining gift cards with EOQ models for the first time
- Investigate the impact of gift cards on all three members of the chain due to inflation.
- Evaluate the impact of third-party gift card sales on order quantity and profit and loss status of chain members.
- Create a two-tier supply chain with a supplier, retailer, and a third party with a gift card in mind
- Provide numerical examples to provide improved models and algorithms.
- Consider several concepts of rules or regulations regarding real-life gift cards.
- Calculate the optimal order of goods at the retailer level and supplier level and the optimal price of each gift card at the retail level for sale to customers and third parties, and the optimal price of gift cards provided to customers by third parties.

The continuation of this article is as follows.

Problem statements and concepts are presented in Section 2. In Section 3, the problem is formulated. Numerical examples are illustrated in Section 4. In Section 5, a sensitivity analysis is performed on some parameters and finally, the conclusions and Some roadmaps for future research are given in Section 6. In this paper, the term 'card' will be used instead of the 'gift card'.

Problem Definition

Fig. 1 shows a schematic view of the proposed problem in which there is a retailer and a third party at the first level and a supplier at the second level. The supplier provides goods needed by the retailer and pays for the goods in Receiving delivery time from the retailer. The retailer often sells products received from the supplier for P_r in cash. But at some times, such as the beginning of the New Year, festivals, etc., they start selling cards. The retailer sells cards either directly to its customers at the $P_r(1-\beta)$ (it gives % β discount) or sells them to a third party at the price of P_{rt}. In this study, it is assumed that the retailer sells all of its cards to its customers and third parties at zero time and below the sale price of its goods. At the beginning of those days (M), it also sells its goods with cards. Printed cards are valid until the end of the retail period (H). Given that the third party is assumed to be larger in size than the retailer, the third party has the ability to sell cards in price P_{tc} to both its customers and the retailer, so that $P_{tc} < P_{rt}$. Due to the fact that the third party is larger than the retailer, not all customers have easy access to it. Depending on the sale price and customer access, customers may purchase cards from both the retailer and a third party. Retailers offer cards for two purposes. For your own use or as a gift to someone else. If the customer has bought the card for personal use, he must wait until M when the goods start to sell, and if he has bought the card for the gift, he has the opportunity to give it as a gift, which is valid until the expiration date of the card.

The demand for cards from retailers depends on two factors. The first and most influential factor is the discount percentage of the card (β) that the higher the discount, the more the number of customers. The next factor is the selling price of the card by a third party (P_{tc}). If the difference between the selling price of the card by a third party and the retailer is significant, potential customers will buy it from a third party. In this model, the customers of the card buyer are divided into three groups as shown in Fig. 2.

The retailer's market size with cash and cards is D and S, respectively and for the third party, it is equal to Z. In fact, retailers have two types of customers: Group D, which supplies goods in cash (Part 1), and Group S, which purchases goods from retailers via cards (Part 2). But depending on the utility function, market size D may be smaller and market size S larger or vice versa (Section 4). Group Z customers are third parties from whom customers receive a card and purchase goods from a retailer (Part 3). But depending on the utility function, market size Z may be smaller and market size S may be larger or vice versa (Section 5). It should be noted that the percentage of customers who buy cards do not go to retail and buy their goods. This number $\alpha_3 s$ is for retail customers and $\lambda_3 z$ is for third-party customers.

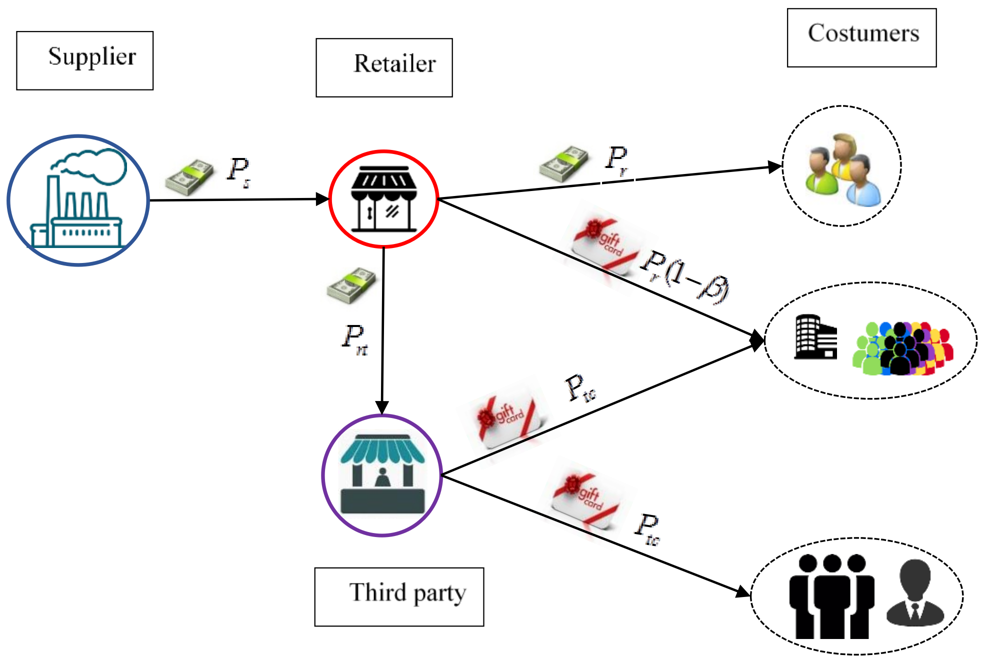


Fig. 1. The conceptual figure of the proposed problem with the sale price of the member

First, third-party consumer preferences are analyzed. A third-party customer can benefit $U_t = \phi - P_{rt} + P_{tc}$ from buying a card from a third party. If $U_t > 0$ they buy a card from a third-party. Otherwise, he leaves the market or buys it from a third party. It is easy to submit a card request from a third-party as follows: $D_t = \left(b^{'}(P_r(1-\beta) - P_{tc})\right)$

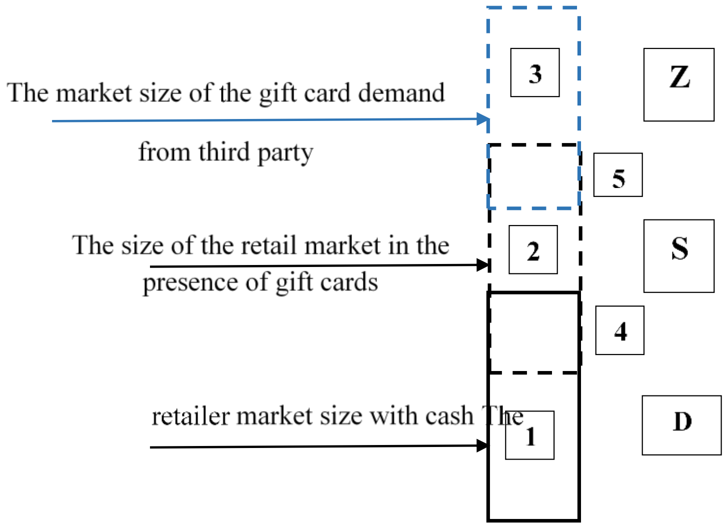


Fig. 2. The retailers final demand in the presence of the card

The aim of this study is to calculate the discount rate of the card by the retailer, the price of the card that is sold by the retailer to a third party and the price of the card that is sold by the third party to its customers, as well as the optimal order amount of the retailer and supplier. Cards are available in two modes of cooperative and non-cooperative state of supply chain members. Dependent variables are the order by the retailer and supplier in time periods, also the sale price of cards by the retailer to the customer and the third party, which are calculated by decision variables. The purpose of calculating these variables is to calculate them separately in the first model, which leads to maximizing the profits of all members of the supply chain, and in the second model, maximizing the profits of the whole chain.

The following are other decision variables and parameters:

Parameters:

P_r Price of items in retailer level
D_r Regular demand rate in retailer level
H Planning horizon
θ Inflation rate
$h^{'}$ Card providing cost
I_e Retailer interest rate earnings per dollar over a specified period.
A_r Retailer ordering cost
I_{hr} Product holding cost in retailer level
P_s Price of items those are selling by supplier
L The time between the release of cards and the start of the sales season
S The sum of card that are sold at The beginning of the sales period which is defined by sH. s is the purchase rate with card which is defined by $a + b\beta$. a and b parameters are constant.
I_h Each product holding cost per time for the supplier
C_s Each items purchasing price for the supplier
F Each delivered goods cost from supplier level to retailer level
A_s Supplier ordering cost
Z Purchase rate of a card from third party which is equal to $b^{'}(P_r(1-\beta) - P_{tc})$ and $b^{'}$ is a parameter and constant

$\alpha_1, \alpha_2, \alpha_3$ will be defined in the text

Decision Variables:

P_{rt} Price of each card that is sold by retailer to third party ($P_{rt} < P_r(1-\beta)$)
P_{tc} Price of each card that is sold by third party to customers ($P_{rt} < P_{tc} < P_r(1-\beta)$)
n_s The amount of goods that are transferred from the supplier level to the retailer level within a specified period of time.
n_r The replenishments number (during the planning horizon.)
β Cards discount percentage

Dependent Variables:

IN_s Suppliers income during whole period of H at the beginning
SE_s Supplier ordering cost in the whole period of Hat the beginning
HO_s Supplier holding cost in the whole period of Hat the beginning
PU_s Supplier purchasing cost in the whole period of Hat the beginning
TE_s Supplier transportation cost in the whole period of Hat the beginning
IN_r Retailer income in the whole period of Hat the beginning
SE_r Retailer ordering cost in the whole period of Hat the beginning
HO_r Retailer holding cost in the whole period of Hat the beginning
PU_r Retailer purchasing cost in the whole period of Hat the beginning
PC_r Retailer profit in the whole period of Hat the beginning
PC_t Third party profit in the whole period of Hat the beginning
PC_s Supplier profit in the whole period of Hat the beginning
PC_{rsi} Profit of the supply chain during whole period of Hat the beginning for *i*, *i = 1,2,3*

Model Development

In this section, first, the costs of echelon1 (third party and retailer) and echelon2 (supplier) are calculated, and then two models are developed in non-cooperating and cooperating modes of echelons. In the following, the proposed model is explained with respect to the interaction between two echelons, the card incentive policy and a dual-channel supply chain.

Costs and incomes of echelon 1

In this section, we get the costs and benefits of echelon1, which includes third-party and retailer.

Third party's costs and incomes

The Market size of third party for selling cards is Z. Since the amount of third party customers' demand for cards depends on the selling price of the card by the retailer and the third party, so the demand for third party sction is equal to $z = a' + (P_r(1-\beta) - P_{tc})b'$ where $a'$and $b'$are constant. The only cost of a third party is the cost of buying cards, which is equal to$(a' + (P_r(1-\beta) - P_{tc})b')HP_{rt}$. And third-party's incomes from selling cards is $(a' + (P_r(1-\beta) - P_{tc})b')HP_{tc}$. As a result, the third-party profit function is as follows.

$$PC_t = \underbrace{\left(a' + \left(P_r(1-\beta) - P_{tc}\right)b'\right)H\left(P_{tc} - P_{rt}\right)}_{\text{Income from saling gift card}} \tag{1}$$

Retailer's costs and incomes

Incomes:

According to Appendix A, the amount of retail revenue in the range of [0,H] is as the following four parts:

$$
\begin{aligned}
IN_r = {} & \underbrace{P_r\left(D_r-\alpha_1\frac{S}{H}\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{Selling product directly}} \\
& +\underbrace{\left((a+\beta b)-\lambda_1\left(a'+b'\left(P_r(1-\beta)-P_{tc}\right)\right)\right)H\left[P_r(1-\beta)-h'\right]e^{\theta L}}_{\text{Selling product with gift card to your customers}} \\
& +\underbrace{\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right]e^{\theta L}}_{\text{Selling gift card to third party}} \\
& +\underbrace{\underbrace{I_eL\left(\left[P_r(1-\beta)-h'\right](S-\lambda_1 Z)+\left[P_{rt}-h'\right]Z\right)}_{I}
+\underbrace{\left(\frac{\left[P_r(1-\beta)-h'\right]\left((\alpha_1+\alpha_2)S-\lambda_1 Z\right)}{H}+\frac{\left[P_{rt}-h'\right](1-\lambda_3)Z}{H}\right)I_e\frac{H^2}{n_r^2}\sum_{i=1}^{n_r}\left((n_r-i)+\frac{1}{2}\right)e^{-i\theta\frac{H}{n_r}}}_{II}
+\underbrace{\left(\frac{\left[P_r(1-\beta)-h'\right]\alpha_3 S+\left[P_{rt}-h'\right]\lambda_3 Z}{H}\right)I_e\frac{H^2}{n_r^2}\sum_{i=1}^{n_r}e^{-i\theta\frac{H}{n_r}}}_{III}}_{\text{The interest earned of selling gift cards before selling products}}
\end{aligned}
\tag{2}
$$

Costs:
Retail costs include the cost of ordering, maintenance, and purchasing goods, which is calculated as in Table 1.

Table 1. All of the retailer's costs

Cost description	Amount of income	
Ordering cost	$\sum_{j=0}^{n_r-1} A_{(jT_r)} = \sum_{j=0}^{n_r-1} A_r e^{-(j\theta T_r)} = A_r\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]$	(3)
Holding cost	$TC_A = \sum_{j=0}^{n_r-1} I_{hr}P_{s(jT_r)}\frac{(D_r+\alpha_2 s+\lambda_2 z){T_r}^2}{2}e^{-(j\theta T_r)} = I_{hr}P_s\frac{\left(D_r+\alpha_2(a+\beta b)+\lambda_2\left(a'+(P_r(1-\beta)-P_{tc})b'\right)\right){T_r}^2}{2}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]$	(4)
Purchasing cost	$IN_s = \sum_{j=0}^{\frac{n_r}{n_s}-1} P_{s(j)}\,(D_r+\alpha_2 s+\lambda_2 z)T_r = \sum_{j=0}^{\frac{n_r}{n_s}-1} P_s e^{-(j\theta T_s)}(D_r+\alpha_2 s+z)T_r = P_s\left(D_r+\alpha_2(a+\beta b)+\lambda_2\left(a'+b'(P_r(1-\beta)-P_{tc})\right)\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{n_sH}{n_r}}-1}\right]$	(5)

Costs and Incomes of echelon 2

The total cost of echelon2, which only includes suppliers, all over the period H includes holding, setup, purchasing, and transportation costs. Calculation of these costs are as what follows:

Holding costs

The supplier's maintenance cost is calculated by multiplying the average inventory of each period by the maintenance cost, however, depending on whether the remaining balance n_r/n_s is zero or not, the average inventory of the last period may vary. In Appendix B these calculations are performed. According to the results of calculations in Appendix B and taking the number of orders and inflation into account, the total maintenance cost of items 1, 1-2 and 2-2 is defined as follows.

Mode 1

$$\begin{aligned} HO_{s1} &= \sum_{j=0}^{n_s-1} I_h C_j \overline{I_j} = \sum_{j=0}^{n_s-1} I_h C_s \frac{D_s T_r}{2}\left(\frac{n_r}{n_s}-1\right) e^{-\left(j\theta \frac{H}{n_s}\right)} \\ &= I_h C_s \frac{D_s H}{2n_r}\left(\frac{n_r}{n_s}-1\right)\left[\frac{e^{-n_s\theta\frac{H}{n_s}}-1}{e^{-\theta\frac{H}{n_s}}-1}\right] = I_h C_s \frac{(D_r+\alpha_2 s+\lambda_2 z)H}{2n_r}\left(\frac{n_r}{n_s}-1\right)\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_s}}-1}\right] \end{aligned} \tag{6}$$

Mode 1-2

$$\begin{aligned} HO_{s2} &= \sum_{j=0}^{n_s-1} I_h C_j \overline{I_j} = \sum_{j=0}^{n_s-2} I_h C_s e^{-\left(j\theta\left[\frac{n_r}{n_s}\right]T_r\right)} \frac{D_s T_r}{2}\left(\left[\frac{n_r}{n_s}\right]-1\right) + I_h C_s \bar{I}_{n_s} e^{-\left(\theta(n_s-1)\left[\frac{n_r}{n_s}\right]T_r\right)} \\ &= I_h C_s \frac{D_s H}{2n_r}\left(\left[\frac{n_r}{n_s}\right]-1\right)\left[\frac{e^{-(n_s-1)\theta\left[\frac{n_r}{n_s}\right]T_r}-1}{e^{-\theta\left[\frac{n_r}{n_s}\right]T_r}-1}\right] \\ &\qquad + I_h C_s \frac{\left(n_r-\left[\frac{n_r}{n_s}\right](n_s-1)\right)D_s T_r}{2} e^{-\left(\theta(n_s-1)\left[\frac{n_r}{n_s}\right]T_r\right)} \\ &= I_h C_s \frac{(D_r+\alpha_2 s+\lambda_2 z)H}{2n_r}\left(\left(\left[\frac{n_r}{n_s}\right]-1\right)\left[\frac{e^{-(n_s-1)\theta\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}}-1}{e^{-\theta\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}}-1}\right]\right. \\ &\qquad \left. + \left(n_r-\left[\frac{n_r}{n_s}\right](n_s-1)\right) e^{-\left(\theta(n_s-1)\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}\right)}\right) \end{aligned} \tag{7}$$

Mode 2-2

$$\begin{aligned} HO_{s2} &= \sum_{j=0}^{n_s-1} I_h C_j \overline{I_j} = \sum_{j=0}^{n_s-2} I_h C_s e^{-\left(j\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r\right)} \frac{D_s T_r}{2}\left[\frac{n_r}{n_s}\right] + I_h C_s \bar{I}_{n_s} e^{-\left(\theta(n_s-1)\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r\right)} \\ &= I_h C_s \frac{D_s H}{2n_r}\left[\frac{n_r}{n_s}\right]\left[\frac{e^{-(n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r}-1}{e^{-\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r}-1}\right] + I_h C_s \frac{\left(n_r-\left(\left[\frac{n_r}{n_s}\right]+1\right)(n_s-1)\right)D_s T_r}{2} e^{-\left(\theta(n_s-1)\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r\right)} \\ &= I_h C_s \frac{(D_r+\alpha_2 s+\lambda_2 z)H}{2n_r}\left(\left[\frac{n_r}{n_s}\right]\left[\frac{e^{-(n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)\frac{H}{n_r}}-1}{e^{-\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)\frac{H}{n_r}}-1}\right]\right. \\ &\qquad \left. + \left(n_r-\left(\left[\frac{n_r}{n_s}\right]+1\right)(n_s-1)\right) e^{-\left(\theta(n_s-1)\left(\left[\frac{n_r}{n_s}\right]+1\right)\frac{H}{n_r}\right)}\right) \end{aligned} \tag{8}$$

Setup cost

The number of orders in the first case and second case are equal and one number less than the number in the third case. Also, during H period, the supplier orders N_s times and the whole ordering cost at beginning of the period in presence of inflation is:
For the first and second states:

$$SE_{s1} = \sum_{j=0}^{n_s-1} A_{(jT_s)} = \sum_{j=0}^{n_s-1} A_s e^{-\left(j\theta \frac{H}{n_s}\right)} = A_s \left[\frac{e^{-\theta H} - 1}{e^{-\theta \frac{H}{n_s}} - 1}\right] \tag{9}$$

$$SE_{s2} = \sum_{j=0}^{n_s-1} A_{(jT_s)} = \sum_{j=0}^{n_s-1} A_s e^{-\left(j\theta \left[\frac{n_r}{n_s}\right] T_r\right)} = A_s \left[\frac{e^{-n_s\theta \left[\frac{n_r}{n_s}\right] T_r} - 1}{e^{-\theta \left[\frac{n_r}{n_s}\right] T_r} - 1}\right] \tag{10}$$

For the third mode

$$SE_{s3} = \sum_{j=0}^{n_s-1} A_{s(jT_s)} = \sum_{j=0}^{n_s-1} A_s e^{-\left(j\theta \left(\left[\frac{n_r}{n_s}\right]+1\right) T_r\right)} = A_s \left[\frac{e^{-n_s\theta \left(\left[\frac{n_r}{n_s}\right]+1\right) T_r} - 1}{e^{-\theta \left(\left[\frac{n_r}{n_s}\right]+1\right) T_r} - 1}\right] \tag{11}$$

Transport cost

With respect to N_r retailer ordering times, the total transportation cost in H period can be calculated by Eq. 12:

$$TE_s = \sum_{j=0}^{n_r-1} F_{(jT_r)} = \sum_{j=0}^{n_r-1} F e^{-(j\theta T_r)} = F \left[\frac{e^{-n_r\theta T_r} - 1}{e^{-\theta T_r} - 1}\right] = F \left[\frac{e^{-\theta H} - 1}{e^{-\theta \frac{H}{n_r}} - 1}\right] \tag{12}$$

Purchasing cost

The cost of purchasing is:
For the first mode

$$\begin{aligned} PU_s &= \sum_{j=0}^{n_s-1} C_{s(j)}\, D_s T_s = \sum_{j=0}^{n_s-1} C_s e^{-(j\theta T_s)} D_s T_s = C_s D_s T_s \left[\frac{e^{-n_s\theta T_s} - 1}{e^{-\theta T_s} - 1}\right] \\ &= C_s (D_r + \alpha_2 s + \lambda_2 z) \frac{H}{n_s} \left[\frac{e^{-\theta H} - 1}{e^{-\theta \frac{H}{n_s}} - 1}\right] \end{aligned} \tag{13}$$

For the second mode

$$\begin{aligned} PU_{s2} &= \left(\sum_{j=0}^{n_s-2} C_{s(j)}\, D_s \left[\frac{n_r}{n_s}\right] T_r\right) + C_{s(n_s)} D_s \left(n_r - \left[\frac{n_r}{n_s}\right](n_s - 1)\right) T_r e^{-\left((n_s-1)\theta \left[\frac{n_r}{n_s}\right] T_r\right)} \\ &= C_s (D_r + \alpha_2 s + \lambda_2 z) \frac{H}{n_r} \left(\left[\frac{n_r}{n_s}\right] \left[\frac{e^{-(n_s-1)\theta \left[\frac{n_r}{n_s}\right] T_r} - 1}{e^{-\theta \left[\frac{n_r}{n_s}\right] T_r} - 1}\right] \right. \\ &\quad \left. + e^{-\left((n_s-1)\theta \left[\frac{n_r}{n_s}\right] T_r\right)} \left(n_r - \left[\frac{n_r}{n_s}\right](n_s - 1)\right) \right) \end{aligned} \tag{14}$$

For the third mode

$$PU_{s3} = \left(\sum_{j=0}^{n_s-2} C_{s(j)}\, D_s \left(\left[\frac{n_r}{n_s}\right]+1\right) T_r\right) + C_{s(n_s)} D_s \left(n_r - \left(\left[\frac{n_r}{n_s}\right]+1\right)(n_s-1)\right) T_r e^{-\left((n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r\right)}$$
$$= C_s(D_r+\alpha_2 s+\lambda_2 z)\frac{H}{n_r}\left(\begin{array}{c}\left(\left[\frac{n_r}{n_s}\right]+1\right)\left[\dfrac{e^{-(n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r}-1}{e^{-\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r}-1}\right] \\ +e^{-\left((n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r\right)}\left(n_r-\left(\left[\frac{n_r}{n_s}\right]+1\right)(n_s-1)\right)\end{array}\right) \tag{15}$$

Therefore, the value of $PU_s = \begin{cases} PU_{s1} if\, \frac{n_r}{n_s} = \left[\frac{n_r}{n_s}\right] \\ PU_{s2} if\, \frac{n_r}{n_s} \neq \left[\frac{n_r}{n_s}\right] Rounddown(\left[\frac{n_r}{n_s}\right]) \\ PU_{s3} if\, \frac{n_r}{n_s} \neq \left[\frac{n_r}{n_s}\right] \quad Roundup(\left[\frac{n_r}{n_s}\right]) \end{cases}$ is equal to:

$$PU_s = \begin{cases} PU_{s1} if\, \dfrac{n_r}{n_s} = \left[\dfrac{n_r}{n_s}\right] \\ PU_{s2} if\, \dfrac{n_r}{n_s} \neq \left[\dfrac{n_r}{n_s}\right] Rounddown(\left[\dfrac{n_r}{n_s}\right]) \\ PU_{s3} if\, \dfrac{n_r}{n_s} \neq \left[\dfrac{n_r}{n_s}\right] \quad Roundup(\left[\dfrac{n_r}{n_s}\right]) \end{cases} \tag{16}$$

Supplier's Income

Since the supplier only sends goods to one retailer, the demand in supplier level is equal to the demand in retailer. The rate of demand by cash in retailer level is $(D_r - \alpha_1 s)$ and by card is $(\alpha_1 + \alpha_2)s + \lambda_2 z$. Hence the demand rate is $(D_r + \alpha_2 s + \lambda_2 z)$. With these interpretations, the amount of supplier income with the presence of inflation throughout the *H* period at zero is as follows:

$$IN_s = \sum_{j=0}^{n_r-1} P_{s(j)}\, D_s T_r = \sum_{j=0}^{n_r-1} P_s e^{-(j\theta T_r)} D_s T_r$$
$$= P_s D_s T_r \left[\frac{e^{-n_r\theta T_r}-1}{e^{-\theta T_r}-1}\right] = P_s(D_r+\alpha_2 s+\lambda_2 z)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right] \tag{17}$$

Non-integrated model (Model 1) and integrated all member of supply chain model (Model 2)

The benefit function of each echelon of the chain (retailer and supplier & third party) in this part, is presented in a separate section. Indeed, in Model 1 the goal of each member of the supply chain is maximizing its profits without considering other members, but in Model 2 a profit function according to the incomes and costs of all members of the supply chain (suppliers, retailers and third-parties) has been created.

Model 1

The level 1 profit function according to the calculated revenues and expenses is equal to:

$$
\begin{aligned}
PC_{rt1} = {} & \underbrace{P_r D_r \frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right] - P_r\alpha_1\frac{(a+\beta b)}{H}\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{selling product directly}} \\
& + \underbrace{\left[(a+\beta b)-\lambda_1\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\right]H\left[P_r(1-\beta)-h'\right]e^{\theta L}}_{\text{selling product with gift card to your customers}} \\
& + \underbrace{\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right]e^{\theta L}}_{\text{selling gift card to third party}} \\
& + \underbrace{I_e L\left((a+\beta b)\left[P_r(1-\beta)-h'\right]+\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\left[P_{rt}-h'\right]\right)H}_{\text{the interest earned of selling gift cards before selling products (I)}} \\
& + \underbrace{\left(\begin{array}{l}(a+\beta b)H\left[P_r(1-\beta)-h'\right](\alpha_1+\alpha_2)\\ +\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right](1-\lambda_1)\end{array}\right)I_e\frac{H^2}{n_r^2}\sum_{i=1}^{n_r}\left((n_r-i)+\frac{1}{2}\right)e^{-i\theta\frac{H}{n_r}}}_{\text{the interest earned of selling gift cards before selling products (II)}} \\
& + \underbrace{\left(\begin{array}{l}(a+\beta b)H\left[P_r(1-\beta)-h'\right]\alpha_3\\ +\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right]\lambda_3\end{array}\right)I_e\frac{H^2}{n_r^2}\sum_{i=1}^{n_r}e^{-i\theta\frac{H}{n_r}}}_{\text{the interest earned of selling gift cards before selling products (III)}} \\
& - \left[\begin{array}{l}\underbrace{A_r\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{Ordering cost}} + \underbrace{I_{hr}P_s\frac{\left(D_r+\alpha_2(a+\beta b)+\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\right)T_r^{2}}{2}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{holding cost}} \\ + \underbrace{P_s\left(D_r+\alpha_2(a+\beta b)+\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{n_s H}{n_r}}-1}\right]}_{\text{Purchasing cost}}\end{array}\right] \\
& + \underbrace{\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\left(P_{tc}-P_{rt}\right)}_{\text{third party's Income from saling gift card}}
\end{aligned}
\tag{18}
$$

And echelon2's profit function according to calculated costs and incomes is:

$$
PC1_s(n_s) = \begin{cases} PC1_{s1}(n_s)\, if\, \dfrac{n_r}{n_s} = \left[\dfrac{n_r}{n_s}\right] \\ PC1_{s2}(n_s)\, if\, \dfrac{n_r}{n_s} \neq \left[\dfrac{n_r}{n_s}\right] RoundDown\left[\dfrac{n_r}{n_s}\right] \\ PC1_{s3}(n_s)\, if\, \dfrac{n_r}{n_s} \neq \left[\dfrac{n_r}{n_s}\right] \quad RoundUp\left[\dfrac{n_r}{n_s}\right] \end{cases} \tag{19}
$$

If n_r is divisible by n_s

$$
\begin{aligned}
PC_{s1}(n_s) = {} & P_s(D_r+\alpha_2 s+\lambda_2 z)\frac{H}{n_s}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_s}}-1}\right] - A_s\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{T_r}{n_s}}-1}\right] - F\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right] \\
& -C_s(D_r+\alpha_2 s+\lambda_2 z)\frac{H}{n_s}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_s}}-1}\right] \\
& - I_h C_s\frac{(D_r+\alpha_2 s+\lambda_2 z)H}{2n_r}\left(\frac{n_r}{n_s}-1\right)\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_s}}-1}\right]
\end{aligned}
\tag{20}
$$

If n_r is not divisible by n_s and $\left[\frac{\boldsymbol{n_r}}{\boldsymbol{n_s}}\right]$ is trend downwards.

$$\begin{aligned}
PC_{s2}(n_s) = {} & P_s(D_r + \alpha_2 s + \lambda_2 z)\frac{H}{n_r}\left[\frac{e^{-\theta H} - 1}{e^{-\theta\frac{H}{n_s}} - 1}\right] - A_s\left[\frac{e^{-n_s\theta\left[\frac{n_r}{n_s}\right]T_r} - 1}{e^{-\theta\left[\frac{n_r}{n_s}\right]T_r} - 1}\right] - F\left[\frac{e^{-\theta H} - 1}{e^{-\theta\frac{H}{n_r}} - 1}\right] \\
& -C_s(D_r + \alpha_2 s + \lambda_2 z)\frac{H}{n_r}\left(\left[\frac{n_r}{n_s}\right]\left[\frac{e^{-(n_s-1)\theta\left[\frac{n_r}{n_s}\right]T_r} - 1}{e^{-\theta\left[\frac{n_r}{n_s}\right]T_r} - 1}\right]\right. \\
& \qquad\left. + e^{-\left((n_s-1)\theta\left[\frac{n_r}{n_s}\right]T_r\right)}\left(n_r - \left[\frac{n_r}{n_s}\right](n_s - 1)\right)\right) \\
& -I_h C_s\frac{(D_r + \alpha_2 s + \lambda_2 z)H}{2n_r}\left(\left(\left[\frac{n_r}{n_s}\right] - 1\right)\left[\frac{e^{-(n_s-1)\theta\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}} - 1}{e^{-\theta\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}} - 1}\right]\right. \\
& \qquad\left. + \left(n_r - \left[\frac{n_r}{n_s}\right](n_s - 1)\right)e^{-\left(\theta(n_s-1)\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}\right)}\right)
\end{aligned} \tag{21}$$

If n_r is not divisible by n_s and $\left[\frac{\boldsymbol{n_r}}{\boldsymbol{n_s}}\right]$ is trend upwards.

$$\begin{aligned}
PC_{s3}(n_s) = {} & P_s(D_r + \alpha_2 s + \lambda_2 z)\frac{H}{n_r}\left[\frac{e^{-\theta H} - 1}{e^{-\theta\frac{H}{n_s}} - 1}\right] - A_s\left[\frac{e^{-n_s\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r} - 1}{e^{-\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r} - 1}\right] - F\left[\frac{e^{-\theta H} - 1}{e^{-\theta\frac{H}{n_r}} - 1}\right] \\
& -C_s(D_r + \alpha_2 s + \lambda_2 z)\frac{H}{n_r}\left(\left(\left[\frac{n_r}{n_s}\right] + 1\right)\left[\frac{e^{-(n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r} - 1}{e^{-\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r} - 1}\right]\right. \\
& \qquad\left. + e^{-\left((n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r\right)}\left(n_r - \left(\left[\frac{n_r}{n_s}\right] + 1\right)(n_s - 1)\right)\right) \\
& -I_h C_s\frac{(D_r + \alpha_2 s + \lambda_2 z)H}{2n_r}\left(\left[\frac{n_r}{n_s}\right]\left[\frac{e^{-(n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)\frac{H}{n_r}} - 1}{e^{-\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)\frac{H}{n_r}} - 1}\right]\right. \\
& \qquad\left. + \left(n_r - \left(\left[\frac{n_r}{n_s}\right] + 1\right)(n_s - 1)\right)e^{-\left(\theta(n_s-1)\left(\left[\frac{n_r}{n_s}\right]+1\right)\frac{H}{n_r}\right)}\right)
\end{aligned} \tag{22}$$

Model 2

According to the calculated incomes and expenses of all members in the supply chain calculated in Appendix C, the profit function of whole supply chain is:

$$PC_{rt2}(n_s, n_r, \beta, P_{rt}, P_{tc}) = \begin{cases} PC_{rt21}(n_s, n_r, \beta, P_{rt}, P_{tc}) if \dfrac{n_r}{n_s} = \left[\dfrac{n_r}{n_s}\right] \\ PC_{rt22}(n_s, n_r, \beta, P_{rt}, P_{tc}) if \dfrac{n_r}{n_s} \neq \left[\dfrac{n_r}{n_s}\right] RoundDown \left[\dfrac{n_r}{n_s}\right] \\ PC_{rt23}(n_s, n_r, \beta, P_{rt}, P_{tc}) if \dfrac{n_r}{n_s} \neq \left[\dfrac{n_r}{n_s}\right] \quad RoundUp \left[\dfrac{n_r}{n_s}\right] \end{cases} \tag{23}$$

Solving Method

In this section, in the first model, Algorithm 1 is presented to obtain the retailer and third-party decision variables, and the second algorithm is provided to obtain the supplier decision variables. In the first model, algorithm 1 is presented to obtain the retailer and third-party decision variables, and Algorithm 2 is proposed to obtain the supplier decision variables. A third algorithm has also been developed to obtain the decision variables of all members of the second model chain. It should be noted that in the first and third algorithms, the Stackelberg approach is also used.

Algorithm1 and 2 for obtaining decision variables of the retailer, the supplier and the third party of model 1

The Stackelberg approach between the third party and retailer is assumed. In the Stackelberg model, one member in the role of follower determines the optimal values of the decision variables at his level, while the other member in the role of leader decides on his strategies based on the best actions of the follower members. In our case, the third party is in the role of follower and the retailer is in the role of leader. The third party first establishes the optimal value of your decision variable and then the retailer optimizes his/her own decision policies based on the optimum reactions of the third party. During the planning horizon (n_r) the replenishment times and Sale price of cards by the retailer to the third-party (P_{rt}) make up the retailer's decision variables. The third party decides about sale price of each card by the third party to the customers (P_{tc}).

Using the Stackelberg approach, a third-party's profit function is optimized according to decision variables, namel yn_r, β and P_r.

Third party's decision variable

The first-order partial derivative of Eq. 1 ($PC_t(P_{tc})$) according to P_{tc} is given by:

$$\frac{\partial PC_t(P_{tc})}{\partial P_{tc}} = Ha^{'} + Hb^{'}P_r(1-\beta) - 2Hb^{'}P_{tc} + HP_{rt}b^{'} \tag{24}$$

The optimum value of P_{tc} is obtained by equating zero, the above equation is obtained as follows.

$$P_{tc} = \frac{P_r(1-\beta) + P_{rt}}{2} + \frac{a^{'}}{2b^{'}} \tag{25}$$

Given that$\frac{\partial^2 PC_t(P_{tc})}{\partial^2 P_{tc}} = -2Hb^{'} < 0$, the P_{tc}value obtained in Eq. 25 is its global value.

Retailer's decision variables

The retailer's profit function is equal to (Appendix D):

$$PC_r(\beta, P_{rt}) = \psi_2\beta + \psi_3\beta^2 + \psi_4 P_{rt} + \psi_6 P_{rt}^2 + \psi_5\beta P_{rt} + \psi_1 \tag{26}$$

Where

$$E_1 = I_e \frac{H^2}{{n_r}^2}\sum_{i=1}^{n_r}\left((n_r - i) + \frac{1}{2}\right) e^{-i\theta\frac{H}{n_r}}$$

$$E_2 = I_e \frac{H^2}{{n_r}^2}\sum_{i=1}^{n_r} e^{-i\theta\frac{H}{n_r}}$$

$$E_3 = \left[I_{hr} P_s \frac{{T_r}^2}{2}\left[\frac{e^{-\theta H} - 1}{e^{-\theta\frac{H}{n_r}} - 1}\right] + P_s \frac{H}{n_r}\left[\frac{e^{-\theta H} - 1}{e^{-\theta\frac{n_s H}{n_r}} - 1}\right]\right]$$

$$E_4 = \left(e^{\theta L} + I_e L + E_1(\alpha_1 + \alpha_2) + \alpha_3 E_2\right)$$

$$E_5 = \left(e^{\theta L} + I_e L + (1 - \lambda_1)E_1 + E_2\lambda_3\right)$$

$$\psi_1 = \left(\begin{array}{l}\left(P_r D_r \frac{H}{n_r} - P_r \alpha_1 \frac{a}{n_r} - A_r\right)\left[\frac{e^{-\theta H} - 1}{e^{-\theta\frac{H}{n_r}} - 1}\right] + (P_r a - h^{'} a)HE_4 - \left(\frac{a^{'}h^{'}}{2} + \frac{b^{'}h^{'}P_r}{2}\right)HE_5 \\ -\left(D_r + \alpha_2 a + \lambda_2\frac{b^{'}P_r}{2} + \lambda_2 a^{'} - \frac{a^{'}}{2}\right)E_3\end{array}\right) \tag{27}$$

$$\psi_2 = -P_r\alpha_1 \frac{b}{n_r}\left[\frac{e^{-\theta H} - 1}{e^{-\theta\frac{H}{n_r}} - 1}\right] + \left((P_r - h^{'})b - P_r a\right)HE_4 + \frac{b^{'}P_r}{2}Hh^{'}E_5 - \alpha_2 bE_3 + \frac{b^{'}P_r}{2}\lambda_2 E_3$$

$$\psi_3 = -bP_r HE_4$$

$$\psi_4 = \left(\frac{b^{'}P_r + a^{'} + h^{'}b^{'}}{2}\right)E_5 + \lambda_2\frac{b^{'}P_{rt}}{2}E_3$$

$$\psi_5 = -\frac{b^{'}P_r}{2}HE_5$$

$$\psi_6 = -\frac{b^{'}}{2}HE_5$$

Concavity:

To calculate a closed form solution, the concavity of the objective function must first be proved

Theorem 1. The objective function $PC_r(\beta, P_{rt})$ is concave.

Proof. To prove the Convection of $PC_r(\beta, P_{rt})$ the Hessian matrix equation (H) is used. As shown in Appendix E, $[\beta, P_{rt}]H\begin{bmatrix}\beta \\ P_{rt}\end{bmatrix}$ is non-positive, therefor, the function is concave.

$$[\beta, P_{rt}]\begin{bmatrix}\frac{\partial^2 PC_r(\beta, n_r, P_{rt})}{\partial^2\beta} & \frac{\partial^2 PC_r(\beta, n_r, P_{rt})}{\partial\beta\partial P_{rt}} \\ \frac{\partial^2 PC_r(\beta, n_r, P_{rt})}{\partial P_{rt}\partial\beta} & \frac{\partial^2 PC_r(\beta, n_r, P_{rt})}{\partial^2 P_{rt}}\end{bmatrix}\begin{bmatrix}\beta \\ P_{rt}\end{bmatrix} = -b^{'}H\left(e^{\theta L} + I_e L + (1 - \lambda_3)E_1 + \lambda_3 E_2\right)P_{rt}^2 < 0 \tag{28}$$

Thus, the local optimum for $PC_r(\beta, P_{rt})$ is a global optimum. Taking the partial derivatives of $PC_r(\beta, P_{rt})$ according to β and P_{rt}, respectively, and setting them equal to zero, gives, as shown in Eqs. (F4) and (F5) in Appendix F:

$$\beta = \frac{\psi_2 2\psi_6 - \psi_4\psi_5}{\psi_5\psi_5 - 2\psi_6 2\psi_3} \tag{29}$$

$$P_{rt} = -\frac{\psi_4 + \psi_5\beta}{2\psi_6} \tag{30}$$

Solution feasibility:
P_n must be positive and β must be between 0 and 1 for a solution to be feasible. $P_{rt} > 0$

According to Eq. 30:

$$P_{rt} = -\frac{\psi_4 + \psi_5\beta}{2\psi_6} \geq 0 \tag{31}$$

$$\frac{\psi_4 + \psi_5\beta}{2\psi_6} \leq 0 \tag{32}$$

Given that $\psi_6$is negative;

$$\psi_4 + \psi_5\beta \geq 0 \tag{33}$$

According to Eq. 27:

$$\begin{aligned}&\left(a' + b'P_r + b'h'\right)\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right) + E\frac{\lambda_2 b'}{2H}\\&-b'P_r\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right)\beta \geq 0\end{aligned} \tag{34}$$

Now if we set β equal to its maximum value, then:

$$\left(a' + b'h'\right)\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right) + E\frac{\lambda_2 b'}{2H} \geq 0$$

The above statement is always positive. $0 < \beta < 1$

In Appendix G, we show that in order to have $0 < \beta < 1$, β satisfies the following condition:

The solution processes:
To determine β^*,P_{rt}^*, P_{tc}^* and n_r^* the following steps should be done. In Fig. 3 summarizing flowchart of these steps is shown.
Step 1) Consider $n_r = 1$ then calculate β by Eq. 29 , *if* $\beta > \beta_1$*then* $\beta = \beta_1$ and obtain $P_{rt}$from Eq. 30, otherwise obtain β and $P_{rt}$from Eqs. 29 and 30, respectively.
Step 2) According to values for P_{rt},β and n_r, calculate the value of function $PC_r(n_r, \beta, P_{rt})$from Eq. 26.
Step 3) Set$n_r = n_r + 1$ then calculate β by Eq. 29 , *if* $\beta > \beta_1$*then* $\beta = \beta_1$ and obtain $P_{rt}$from Eq. 30, otherwise, obtain β and $P_{rt}$from Eqs. 29 and 30, respectively.
Step 4) Calculate the objective function$PC_r(n_r, \beta, P_{rt})$according to the output of Eq. 26 in Step 3.
Step 5) if $PC_r(n_r, \beta, P_{rt}) < PC_r\big((n_r - 1), \beta(n_r - 1), P_{rt}(n_r - 1)\big)$, so $n_r{}^* = n_r - 1$, $\beta^* = \beta(n_r - 1)$ and $P_{rt}^* = P_{rt}(n_r - 1)$ then go to Step 6. Otherwise, go to Step 3. Do until meeting the stop criterion.
Step 6) According to values for $n_r{}^*$, β^* andP_{rt}^*, calculate the value of P_{tc}^* from Eq. 25.
Step 7) According to values in Steps 1 to 6 (values of $n_r{}^*$, β^*,P_{rt}^*, and P_{tc}^*), obtain the dependent variables values.

1. Each cards sales price from retailer to customers can be obtained by $P_r\left(1-\beta^*(n_r^*)\right)$
2. The time between orders in retailer level can be obtained by $\frac{H}{n_r^*}$
3. The order quantity that is calculated in each cycle for the retailer from the supplier $\frac{\left(D_r+\alpha_2(a+\beta^*(n_r^*,n_s^*)b)+\lambda_2\left(a'+b'\left(P_r\left(1-\beta^*(n_r^*,n_s^*)\right)-P_{rt}^*\right)\right)\right)H}{n_r^*}$

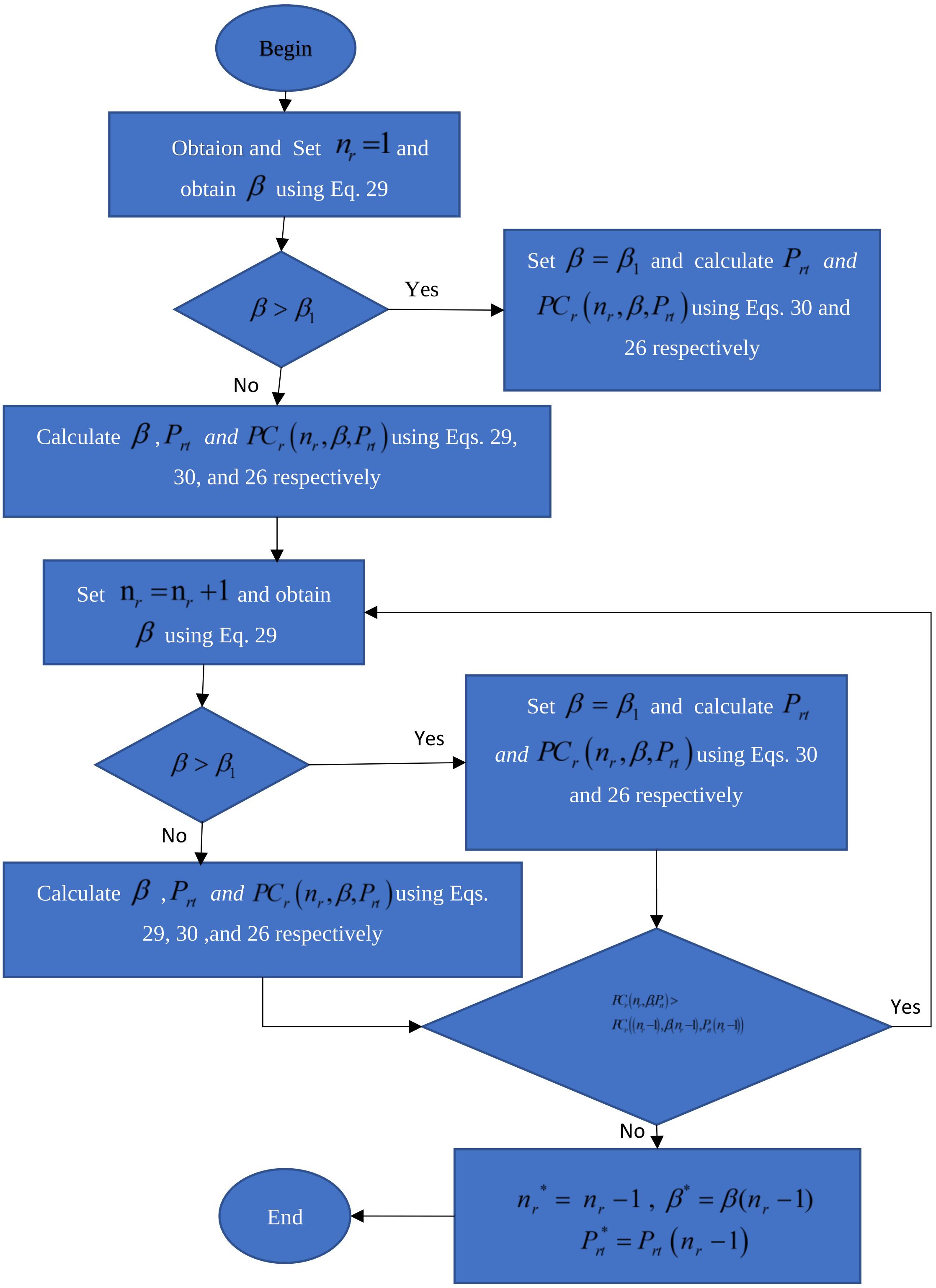


Fig. 3. Flowchart of solution algorithm for retailer’s decision variable

Supplier's decision variable

Since n_s is an integer, to calculate the optimal value of supplier's decision variable we use from the following algorithm.

Algorithm 2 -solving supplier model

Step 1) Consider $n_s = 1$ and calculate $PC_s(n_s)$by Eq. 19.

Step 2) Set $n_s = n_s + 1$, then calculate $PC_s(n_s)$ from Eq. 19.

Step 3) If$PC_s(n_s) < PC_s\big((n_s - 1)\big)$, $n_s{}^* = n_s - 1$ and go to Step 4. Otherwise, go to Step 2. Do it until meeting stop criterion.

Step 4) According to the calculated values by Steps 1 to 3 ($n_s{}^*$values), calculate dependent variables values.

1. The interval between orders in supplier level form $\frac{H}{n_s^*}$
2. In each cycle Order amount for supplier can be calculated by $\frac{\left(D_r+\alpha_2(a+\beta^*(n_r^*,n_s^*)b)+\lambda_2\left(a'+b'\left(P_r\left(1-\beta^*(n_r^*,n_s^*)\right)-P_{rt}^*\right)\right)\right)H}{n_s^*}$

Algorithm 3 for obtaining decision variables of the retailer, supplier, and the third party of mode 2

In this case, the retailer is the leader and the third party also is the follower. The third party first determines the optimal value of the decision variable and then, depending on the best response of the third party, the retailer and supplier optimize their decision policies. Here, during the planning horizon the number of replenishment for retailer (n_r) and supplier (n_s), sale price of cards in retailer level to customers ($P_r(1-\beta)$) and sale price of cards by retailer to third party (P_{rt}) are the retailers decision variables and the third party decide about Selling price of each card by third party to your customers (P_{tc}).

Using the Stackelberg approach, a third party's profit function is optimized according to decision variables namely n_r, β , P_{rt}, and n_s.

Decision variables

As shown in Appendix C, the gain function of the whole chain is:

$$PC_{rt2}(n_s,n_r,\beta,P_{rt},P_{tc}) = \begin{cases} PC_{rt21}(n_s,n_r,\beta,P_{rt},P_{tc}) if\, \frac{n_r}{n_s} = \left[\frac{n_r}{n_s}\right] \\ PC_{rt22}(n_s,n_r,\beta,P_{rt},P_{tc}) if\, \frac{n_r}{n_s} \neq \left[\frac{n_r}{n_s}\right] RoundDown\left[\frac{n_r}{n_s}\right] \\ PC_{rt23}(n_s,n_r,\beta,P_{rt},P_{tc}) if\, \frac{n_r}{n_s} \neq \left[\frac{n_r}{n_s}\right] \quad RoundUp\left[\frac{n_r}{n_s}\right] \end{cases} \tag{24}$$

Concavity

To calculate a closed form solution, the concavity of the objective function must first be proved.

Theorem 1. The objective function $PC_{rt2}(\beta, P_{rt})$is concave.

Given that $\psi_5 = \psi_5' = \psi_5''$, $\psi_6 = \psi_6' = \psi_6''$, and $\psi_3 = \psi_3' = \psi_3''$, as shown in Appendix E, $[\beta, P_{rt}]H\begin{bmatrix}\beta \\ P_{rt}\end{bmatrix} < 0$ and $PC_{rt21}(\beta,P_{rt})$,$PC_{rt22}(\beta,P_{rt})$, and $PC_{rt23}(\beta,P_{rt})$ are concave.

Thus, the local optimum for $PC_{rt21}(\beta,P_{rt})$,$PC_{rt22}(\beta,P_{rt})$, and $PC_{rt23}(\beta,P_{rt})$are a global optimum. By taking the partial derivatives of $PC_{rt2}(\beta,P_{rt})$ according to β and P_{rt} and considering their values as zero, gives, as shown in Eqs. F.4 and F.5 in Appendix F:

$$\beta = -\frac{\psi_2 + \psi_5 P_{rt}}{2\psi_3}$$
$$P_{rt} = -\frac{\psi_2}{\psi_5} - \frac{2\psi_3\beta}{\psi_5}$$

After some substitutions and algebra, we have;

$$\beta = \frac{\psi_2 2\psi_6 - \psi_4\psi_5}{\psi_5\psi_5 - 2\psi_6 2\psi_3} \quad (25)$$
$$P_{rt} = -\frac{\psi_4 + \psi_5\beta}{2\psi_6} \quad (26)$$

The solution process

In order to determine β^*, P_{rt}^*, P_{tc}^*, n_r^*, and n_s^* the following steps should be done.

Step 1) Consider $n_s = 1$ and $n_r = 1$, then calculate β and P_{rt} by Eqs. 25 and 26, respectively.

Step 2) According to calculated values for n_s, n_r, β and P_{rt}, calculate the value of objective function $PC_{rt2}(n_s, n_r, \beta, P_{rt})$ from Eq. 24.

Step 3) Set $n_r = n_r + 1$, then obtain β and P_{rt} from Eqs. 25 and 26, respectively.

Step 4) Calculate the objective function $PC_{rt2}(n_s, n_r, \beta, P_{rt})$ According to the calculated values from Eq. 24 in previous step.

Step 5) If $PC_{rt}(n_s, n_r, \beta, P_{rt}) < PC_{rt}\big(n_s, (n_r - 1), \beta(n_s, n_r - 1), P_{rt}(n_s, n_r - 1)\big)$, then ${n_r}^* = n_r - 1$, $\beta^* = \beta(n_r - 1)$ and $P_{rt}^* = P_{rt}(n_r - 1)$, then go to Step 6. Otherwise, go to Step 3. Do it until meeting stop criterion.

Step 6) According to obtained values for ${n_r}^*$, β^* and $n_r = 1$, obtain the value of P_{tc}^* from Eq. 24.

Step 7) According to the calculated values in Steps 1 to 6 (${n_r}^*$, β^*, P_{rt}^* and P_{tc}^* values), calculate the dependent variables values.

1. Calculate the sale price of cards those are selling by retailer to customers $P_r\big(1 - \beta^*(n_r^*)\big)$
2. The interval between retailer orders can be calculate by P_{rt}

In each cycle calculate the order quantity of retailer to supplier $\frac{\left(D_r + \alpha_2(a + \beta^*(n_r^*, n_s^*)b) + \lambda_2\left(a' + b'(P_r(1 - \beta^*(n_r^*, n_s^*)) - P_{rt}^*)\right)\right)H}{n_r^*}$

Sensitivity Analysis ,Numerical Example, and Managerial Insight

In this section, a numerical example is given for the proposed models and then sensitivity analysis is performed on it and finally, managerial insights are explained.

Case study

As mentioned in the previous sections, cards are one of the sales incentive ways that many companies use to encourage their customers. Retailers have also been encouraged to use independent third parties to sell their cards to increase their sales channel and take advantage of it.

Hayat Market Chain Store Company is one of the largest players in the retail industry in Iran, which is managed by Imtiaz Holding. More than 150 companies in cooperation with this company provide the required goods, and negotiations have been held with some suppliers to issue cards on various occasions, such as the company's founding anniversary, various celebrations, and so on. Cards are sold to various organizations by the company at L time, and products are offered to cardholders within a specified period of time. The proposed models in this paper are implemented and reviewed using the data of this company.

The selected product groups are detergents. The company issued these cards in the last days of October and markets these products through cards. Maintenance costs include the rent per square meter of storage space and other costs that can be considered in the maintenance costs domain. The ordering cost can be obtained based on the costs of the staff.

An average of 20 customers request these products daily. Given that the average number of customers in this branch is equal to 2000 people, this amount of demand is about 1% of the total customers .Therefore, with a 15% increase in the number of customers in recent months, demand is expected to reach 2000 * 15% * 10%. The interest rate paid is based on the loan interest rate and the interest rate that is determined based on the bank's interest rate of the country. Based on our experience in card selling and customers using it, the value of 0.07 is considered, which will always be constant, regardless of the type of supplier. we consider $\boldsymbol{\alpha_1} = \mathbf{0.1}$ since it reached 0.1 of loyal customers because the company has identified its customers according to the number of purchases in previous periods. Since , the value of $\boldsymbol{\alpha_2}$ is 0.85. Other parameters are as follows:

$P_r = 350000$ rials**,** $D_r = 20$ person/day**,** $\boldsymbol{H} = \mathbf{30}$ days**,** $\boldsymbol{\theta} = \mathbf{0.005}$**,** $h' = 10000$ rials**,**$I_e = 0.06\%$**,** $A_r = 3000$ thousand rials**,** $\boldsymbol{I_{hr}} = \mathbf{0.04}$**,** $P_s = 200000$ rials**,** $\boldsymbol{a} = \mathbf{25}$**,** $\boldsymbol{b} = \mathbf{70}$**,** $I_h = 0.015$**,** $C_s = 120000$ rials**,** F = 3000000 rials**,** $A_s = 4500000$ rials**,** $\boldsymbol{a'} = \mathbf{6}$**,** $\boldsymbol{b'} = \mathbf{0.01}\%$**,** $\boldsymbol{L} = \mathbf{30}$ day, $\lambda_1 = 0.1$, $\lambda_2 = 0.7$, $\lambda_3 = 0.2$

Tables 2, 3, 4, and 5 show the results of the proposed model using the problem parameters.

Table 2. Sales prices of goods and cards by members of the chain

Prices		
price	*Model 1*	*Model 2*
C_s	120,000	120,000
P_s	200,000	200,000
P_r	350,000	350,000
$P_r(1-\beta)$	263,410	278,320
P_{rt}	243,180	227,820
P_{tc}	283,300	283,070

Table 3. Demand for goods and cards in both models

Demand for goods and cards		
Demand each section	*Model 1*	*Model 2*
Direct sales demand	533	542
selling product with GF	645	547
selling GF to third party	120	166

Table 4. Chain profit of members of both models

Profit of members		
Profit	***Model 1***	***Model 2***
Retailer	168,600,000	170,780,000
Supplier	66,359,800	58,103,000
Third party	4,828,000	9,160,200
The whole chain	239,787,800	238,043,200

Table 5. Number of times the retailer and manufacturer order in both models

Number of orders	*Model 1*	*Model 2*
n_r	*7*	*5*
n_s	*2*	*1*

Several sensitivity analyzes have been performed to obtain some managerial insights into some of the original model parameters for the second example (the parameters of this example are given in Table 6). To achieve this goal, the parameters D_r, α_3, I_{hr}, H, a, b, $\boldsymbol{A_r}$, $\boldsymbol{A_s}$, θ, P_r, P_s, $h^{'}$ and F change on four levels. In Table 7 the effects of the changes are shown and the following results are obtained.

The results of the sensitivity analysis show that the increase in the prepaid period of the card money leads to an increase in the profit of the entire supply chain and increases the discount percentage of each card. Accordingly, the retailer tries to present the card as soon as possible before the sale begins. Therefore, it is better for managers to reduce the time of card sales to companies, institutions and customers when using the card. In fact, the card discount depends on the time of sale. They can also inform customers before selling cards, which encourages them to buy cards sooner. For example, the sales team can use a step-by-step method to persuade more customers to buy cards sooner and increase store profits.

Table 6. Parameters of the second example

Parameter	Value	Parameter	value	Parameter	value
$\boldsymbol{P_r}$	500000	$\boldsymbol{h^{'}}$	12000	$\boldsymbol{P_s}$	280000
$\boldsymbol{D_r}$	35	$\boldsymbol{I_e}$	0.06%	$\boldsymbol{a}$	6
$\boldsymbol{H}$	30	$\boldsymbol{A_r}$	2800000	$\boldsymbol{b}$	100
$\boldsymbol{\theta}$	0. 6%	$\boldsymbol{I_{hr}}$	0.035	$\boldsymbol{I_h}$	0.017
$\boldsymbol{C_s}$	180000	$\boldsymbol{L}$	25	λ_3	0.2
$\boldsymbol{F}$	3500000	λ_1	0.1	α_1	0.1
$\boldsymbol{A_s}$	5000000	λ_2	0.7	α_2	0.85
α_3	0.05	$\boldsymbol{a^{'}}$	5	$\boldsymbol{b^{'}}$	0.012%

The analysis shows that increasing the card prepayment period increases the supply chain profit but does not necessarily increase the card discount. In fact, this value will vary according to whether the $\frac{\partial \beta}{\partial L}$ is positive or negative. That is, if it is $\frac{\partial \beta}{\partial L} > 0$, increasing L leads to an increase in β, but if $\frac{\partial \beta}{\partial L} < 0$, increasing L leads to a decreasing β.

When the third-party is smaller in size than the retailer, that is, the amount of cards sold by the third party is less than that of retailer, an increase in L increasesβ. But if the third party is larger in size than the retailer, that is, the amount of cards sold by the third party is more than that of the retailer, an increase in L decrease β.

According to the research results, the retailer should sell his cards as soon as possible and before the start of the sales time. Accordingly, it is suggested that when companies have a policy of using cards, their sales team try to sell cards as soon as possible and even offer discounts to improve sales. This means that the discount rate of the card depends on the time of sale, and the longer the time interval between the sale of the card and the start of the sale of goods, the higher the amount of the retailer's discount. In addition, sales agents can inform customers before the start of the card sales time, thus encouraging them to buy cards in a timely manner. For example, the use of step-by-step discount tables relative to the time of sale by the company's sales team can lead to the faster sale of cards and increase the cumulative profit by encouraging the customer.

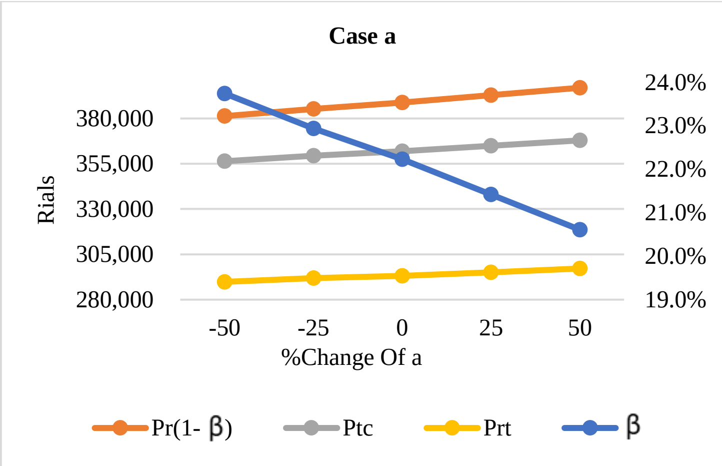
By increasing the received interest rate by the retailer (Ie), the chain revenue and discount of the card increase and lead to a decrease in the sale price of each card by the retailer to a third party. One reason for this is the increase in retailers' profits through the sale of cards. Therefore, Managers when using the card policy can increase the discount rate of cards as well as the chain profit by consulting with banks and financial institutions.

Increasing the issuing cost of cards ($h^{'}$) reduces the income of chain member and their ability to discount any card. This means that to reduce the issuing cost of a card and consequently increase the discount on the card and the profits of chain members, retailers need to work with manufacturers who produce cards at a lower cost than others or cheaper materials to produce cards.

Table 7. The sensitivity analysis of the first example

Parameter	%Changes	Optimal values					% Changes in
		n_r^*	n_s^*	β	P_{rt}	P_{tc}	$TP_{r\&t}(\beta^*, n_s^*, n_r^*, P_{rt}^*, P_{tc}^*)$
D_r	-50	8	2	22.1%	293,861	362,455	371,297,582
	-25	8	2	22.1%	293,861	362,455	423,093,054
	+25	9	3	22.2%	293,138	361,832	520,911,350
	+50	9	3	22.2%	293,138	361,832	573,073,306
P_r	-50	NA	NA	NA	NA	NA	NA
	-25	6	2	4.9%	280,000	339,231	203,327,100
	+25	8	2	24.5%	334,591	424,013	712,331,613
	+50	9	3	25.6%	377,745	488,743	960,429,925
H	-50	5	1	18.4%	315,236	382,454	195,832,207
	-25	6	2	20.1%	305,270	373,104	320,015,558
	+25	12	3	24.2%	281,609	351,099	660,316,510
	+50	15	3	22.3%	280,000	355,139	876,515,268
θ	-50	8	2	20.3%	305,760	373,009	443,021,319
	-25	8	2	21.2%	299,686	367,623	458,564,237
	+25	9	3	23.1%	287,555	356,874	486,399,030
	+50	9	3	23.9%	282,209	352,124	504,854,906
h'	-50	10	2	22.8%	288,756	358,260	482,010,800
	-25	10	2	22.5%	290,947	360,046	476,665,452
	+25	10	2	22.0%	295,329	363,619	466,098,362
	+50	10	2	21.6%	298,242	366,027	464,427,935
A_r	-50	12	2	22.4%	291,641	360,544	475,958,964
	-25	10	2	22.3%	292,547	361,324	474,778,455
	+25	9	3	22.1%	293,861	362,455	466,528,323
	+50	8	2	22.0%	294,766	363,234	467,025,955
I_e	-50	8	2	20.2%	305,042	372,962	423,201,260
	-25	8	2	21.2%	299,203	367,467	448,958,174
	+25	10	2	23.1%	288,260	357,269	497,457,777
	+50	10	2	23.9%	283,764	353,074	523,683,006
I_{hr}	-50	8	2	22.5%	291,324	360,316	492,835,198
	-25	8	2	22.2%	293,045	361,775	485,098,407
	+25	12	2	22.1%	293,796	362,382	462,080,063
	+50	12	2	22.1%	294,343	362,840	453,426,105
P_s	-50	9	3	29.8%	238,873	315,852	499,795,762
	-25	8	2	25.9%	266,820	339,543	492,603,469
	+25	8	2	7.2%	350,000	427,856	393,083,324
	+50	NA					
a	-50	8	2	23.7%	289,815	356,386	449,046,074
	-25	8	2	22.9%	291,838	359,421	461,838,540
	+25	10	2	21.4%	295,161	364,867	484,698,925
	+50	10	2	20.6%	297,183	367,901	498,294,061
b	-50	8	2	13.8%	315,268	393,987	366,521,140
	-25	10	2	19.5%	300,332	372,162	416,965,584
	+25	10	3	23.7%	289,413	356,245	528,665,297
	+50	12	3	24.8%	286,399	352,101	586,670,575
a'	-50	10	2	22.6%	281,900	344,976	463,327,068
	-25	10	2	22.4%	287,519	353,404	467,243,546
	+25	10	2	22.1%	298,757	370,261	475,680,348
b'	-50	8	2	23.9%	310,282	387,087	461,573,362
	-25	8	2	23.0%	298,557	369,499	467,843,257
	+25	10	2	21.3%	291,314	359,096	478,932,461
	+50	10	2	20.3%	290,975	358,588	486,915,102
A_s	-50	9	3	22.2%	293,138	361,832	477,988,146
	-25	9	3	22.2%	293,138	361,832	473,368,769
	+25	10	2	22.2%	293,138	361,832	467,804,981
	+50	10	2	22.2%	293,138	361,832	464,248,657
L	-50	8	2	20.8%	302,217	369,864	438,092,084
	-25	8	2	21.5%	297,969	366,099	456,138,801
	+25	10	2	22.8%	289,187	358,326	490,853,579
	+50	10	2	23.4%	285,365	354,933	511,097,595
C_s	-50	10	2	22.2%	293,138	361,832	612,412,511
	-25	10	2	22.2%	293,138	361,832	541,886,908
	+25	12	3	22.2%	293,138	361,832	398,633,864
	+50	12	3	22.2%	293,138	361,832	328,518,335

Increasing the product sale price increases the discount percentage of the card and the profit of the chain. In other words, the higher the profit margin of the product, the more power the retailer has to offer discounts to customers. Increasing the discount leads to increasing demand and consequently increases the chain's profit. Therefore, retailers need to negotiate with suppliers for more profit margins in order to offer more discounts on cards and increase supply chain members' profits.



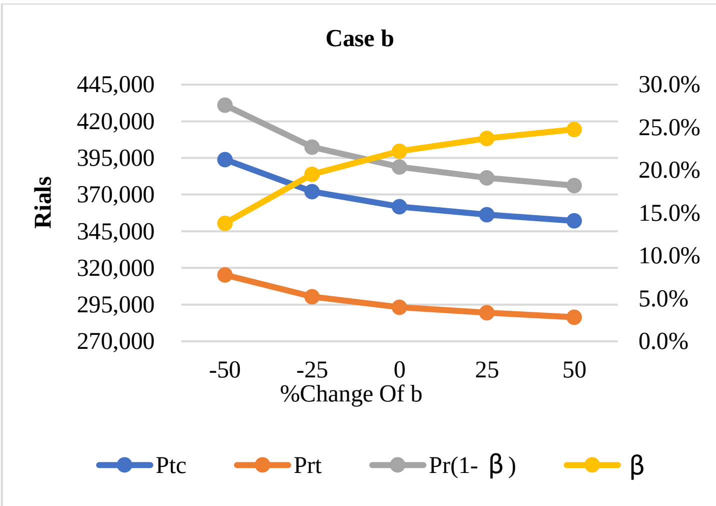


Fig. 4. The discount percentage on cards and the Card prices along with the card purchase rate parameters (a and b) are different.

In the left diagram of Fig. 4, increasing the parameter *a* leads to a decrease in the discount rate of the card. In fact, when the card's demand increases, the discounts given to the card decrease. This means that in the days when receiving and giving cards is high, we can succeed even with a small discount. Also, as parameter *a* increases, the selling price of the retailer card to its customers and third parties, as well as the selling price of the card by the third party, increases. In fact, we can sell cards at a higher price and increase the profit of the whole chain.

But the right diagram of Fig. 4 shows exactly the opposite of the previous case. That is, with increasing the amount of parameter b, the amount of discount also grows up. In this way, if the sensitivity of customers to the amount of discount is considerable, to attract customers, this model will apply more discounts and the selling price of cards will be reduced. It should be noted that although with increasing the amount of b, the amount of discount increases, the slope of this increase is decreasing.

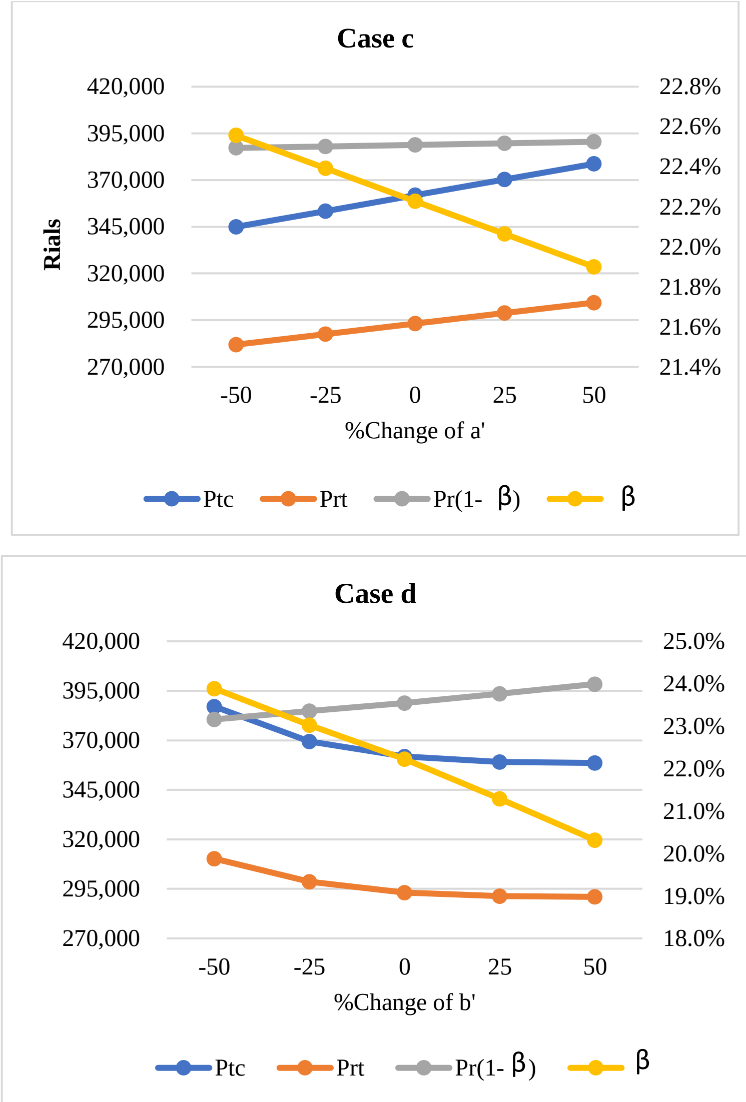


Fig.5. shows that the percentage of discount on each card and the price of the card differ from the purchase rate parameters with the card. (a՝ and b՝).

In the left diagram of Fig. 5, by parameter increasing the "a'" the discount rate for a card decreases. Indeed, when the demand for cards increases - whether they are retailer's customers or third-party customers - the discount on the card decreases. This means that if we use the card strategy on the days when gift giving and receiving is hot (a' is bigger than b'), we can be successful in the market even with less discount on our card. Also, as the parameter 'a' increases, the selling price of the retail card to its customers and third parties, as well as the selling price of the card by the third party, increases. In fact, we can sell cards at a higher price and increase the profit of the whole chain. In fact, the behavior of Case "a" and Case "c" are similar. But looking at the right diagram, we see that the behavior of Case "b" and Case "d" are different. That is, by increasing the amount of parameter b ', the amount of discount decreases, unlike case b. Note that the graph on the right of Fig. 5 shows that Ptc is higher than Pr (1-β), while b 'is reduced by 50%. In fact, if third-party customers who are sensitive to the price of the card have a significant reduction, the selling price of the third party may be higher than the retail price.

The influence of two parameters, L and I_e , on cards and supply chain profitability is discussed. the effect of both parameters can be seen in Table 3. According to Table 8, if a retailer sells a card 50% sooner and then invests funds in projects with a 50% interest rate more than usual, the profit will grow up to 19.3.

Table 8. The effects of interest rates and the time interval between card sales and the start time of product sales on supply chain profits

Changes % in parameter L	Changes % in parameter I_e			
	-%50	-%25	+%25	+%50
-%50	-17.6%	-12.3%	-2.3%	2.9%
-%25	-14.0%	-8.6%	1.5%	6.9%
+%25	-6.3%	-1.5%	9.8%	15.4%
+%50	-2.8%	2.7%	14.2%	19.3%

Conclusion and Future Research

This research is divided into two inventory models by considering the card and a two-channel supply chain with the optimal order quantity policy of ordinary goods in the inventory control system of retailers, third parties and suppliers in two situations of cooperation and non-cooperation of members This paper provides relatively limited academic knowledge on how product-specific cards affect EOQ models and the dual-channel supply chain with conventional products. A number of numerical tests have been designed and performed to confirm and validate the proposed model for the optimal solution. The results show that the use of cards in the case of economic order models increases the demand for retail and on the other hand attracts more customers and better brand expansion. Each model has a specific solution approach and the convexity of all objective functions after extraction to solve this approach is proved and an optimal solution is created for each model. Sensitivity analysis was used to obtain the main factors of the model.

Partial and multi-product backup orders, decay rates, inclusion, and other pricing policies for discount plans can be another area for future work.

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Appendices

Appendix A: Retailer's income calculation

Retailer incomes contain four parts: 1- selling products directly 2- selling products with gift cards to your customers. 3- selling gift cards to third-parties 4- the interest earned from selling gift cards before selling products. Income from selling products directly occurs in intervals [0,H] and due to the number of replenishments, the Time value of money is calculated in Eq. 1.

$$\sum_{j=0}^{n_r-1} P_{r(j)}\,(D_r-\alpha_1 s)T_r=\sum_{j=0}^{n_r-1} P_r e^{-(j\theta T_r)}(D_r-\alpha_1 s)T_r=P_r(D_r-\alpha_1 s)T_r\left[\frac{e^{-n_r\theta T_r}-1}{e^{-\theta T_r}-1}\right]$$
$$=P_r\left(D_r-\alpha_1\frac{S}{H}\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right] \tag{A.1}$$

Note that since $\alpha_1 S$numbers of buyers of gift card are former customers of retailer, the amount of demand with money is $(D_r-\alpha_1 s)$. As mentioned before, retailer discounts βpercent to each gift card and noting that the cost of issuing each gift card ish', therefore the obtained money of selling each gift card is $P_r(1-\beta)-h'$.

On the other hand, the retailer's market size for selling gift cards to its customers is equal to S. Since the number of customers demand to buy a gift card depends on its discount percentage and also the selling price of the gift card by a third party, therefore, its demand is equal to $\Big((a+\beta b)-\lambda_1\big(a'+b'(P_r(1-\beta)-P_{tc})\big)\Big)H$, where $\lambda_1\big(a'+b'(P_r(1-\beta)-P_{tc})\big)H$is the number of retailer and third party customers who buy their gift card from third party. So the obtained money by retailer from selling gift card to your customers at zero instance is $\Big((a+\beta b)-\lambda_1\big(a'+b'(P_r(1-\beta)-P_{tc})\big)\Big)H[P_r(1-\beta)-h']$ where this amount is due to the time value of money at the beginning of the sales period with a gift card (at the moment L) is $\Big((a+\beta b)-\lambda_1\Big(a'+b'(P_r(1-\beta)-P_{tc})\Big)\Big)H\big[P_r(1-\beta)-h'\big]e^{\theta L}$that it is considered in the second part of retailer's income

Given that the incomes from selling each gift card to third party is $[P_{rt}-h']$and the entire third party gift card is provided by the retailer, Therefore, the retailer's incomes at moment L will be $(a'+(P_r(1-\beta)-P_{tc})b')H\big[P_{rt}-h'\big]e^{\theta L}$

The total money of the sold gift card is available to retailer in [0,L], therefore, the amount of interest is $I_eL([P_r(1-\beta)-h'](S-\lambda_1 Z)+[P_{rt}-h']Z)$ at L instance. However, as it is illustrated in Fig. A.1 in first range $\frac{H}{n_r}$, *nr - 1* rectangles with area $I_e([P_r(1-\beta)-h'](s-\lambda_1 z)+[P_{rt}-h']z)\frac{H^2}{{n_r}^2}$ and a triangle with area$I_e([P_r(1-\beta)-h'](s-\lambda_1 z)+[P_{rt}-h']z)\frac{H^2}{2{n_r}^2}$, in second range $\frac{H}{n_r}$, *nr - 2* rectangles and a triangle with the same area and in $n_r th$ range $\frac{H}{n_r}$, there is only one triangle with the same area. So the amount of received interest in this period with respect to the available average money for i*th* period is $I_e([P_r(1-\beta)-h'](s-\lambda_1 z)+[P_{rt}-h']z)\frac{H^2}{{n_r}^2}\Big((n_r-i)+\frac{1}{2}\Big)$, where moving each period to L instance, the total received interest is $\sum_{i=1}^{n_r} I_e([P_r(1-\beta)-h'](s-\lambda_1 z)+[P_{rt}-h']z)\frac{H^2}{{n_r}^2}\Big((n_r-i)+\frac{1}{2}\Big)e^{-i\theta\frac{H}{n_r}}$. With respect to this point that in every period $\alpha_3 s+\lambda_3 z$demand is in our possession, so the received interest is $I_e([P_r(1-\beta)-h']\alpha_3 s+[P_{rt}-h']\lambda_3 z)\frac{H^2}{{n_r}^2}$ in each period that by moving all periods to L instance, the total received money is equal to $I_e\big(\big[P_r(1-\beta)-h'\big]\alpha_3 s+\big[P_{rt}-h'\big]\lambda_3 z\big)\frac{H^2}{{n_r}^2}\sum_{i=1}^{n_r} e^{-i\theta\frac{H}{n_r}}$. Thus, the total received money of selling gift card at the beginning of selling by gift card (at L instance) is equal to

$$\underbrace{I_eL\Big((a+\beta b)\big[P_r(1-\beta)-h'\big]+\big(a'+(P_r(1-\beta)-P_{tc})b'\big)\big[P_{rt}-h'\big]\Big)H}_{\text{the interest earned of selling gift cards before selling products (I)}}$$
$$+\underbrace{\begin{pmatrix}(a+\beta b)H\big[P_r(1-\beta)-h'\big](\alpha_1+\alpha_2)\\+\big(a'+(P_r(1-\beta)-P_{tc})b'\big)H\big[P_{rt}-h'\big](1-\lambda_1)\end{pmatrix}I_e\frac{H^2}{{n_r}^2}\sum_{i=1}^{n_r}\Big((n_r-i)+\frac{1}{2}\Big)e^{-i\theta\frac{H}{n_r}}}_{\text{the interest earned of selling gift cards before selling products (II)}}$$
$$+\underbrace{\begin{pmatrix}(a+\beta b)H\big[P_r(1-\beta)-h'\big]\alpha_3\\+\big(a'+(P_r(1-\beta)-P_{tc})b'\big)H\big[P_{rt}-h'\big]\lambda_3\end{pmatrix}I_e\frac{H^2}{{n_r}^2}\sum_{i=1}^{n_r}e^{-i\theta\frac{H}{n_r}}}_{\text{the interest earned of selling gift cards before selling products (III)}}$$

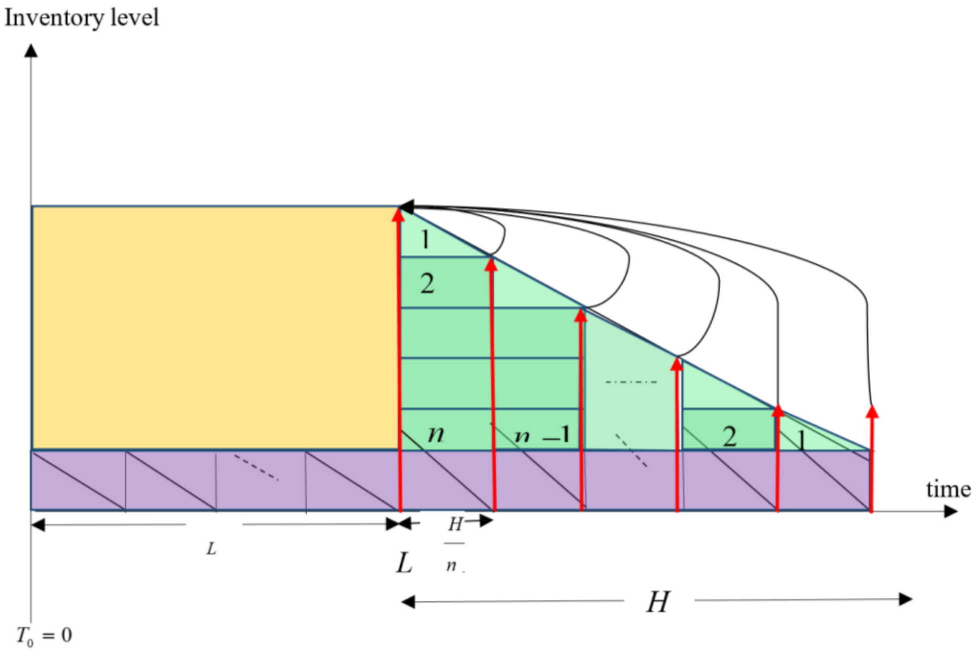

Fig. A.1. interest earned from selling gift card at *L* moment.

Therefore, retailer income is calculated as what follows:

$$
\begin{aligned}
IN_r = & \underbrace{P_r\left(D_r - \alpha_1 \frac{S}{H}\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta \frac{H}{n_r}}-1}\right]}_{\text{Selling product directly}} \\
& + \underbrace{\left((a+\beta b)-\lambda_1\left(a' + b'\left(P_r(1-\beta)-P_{tc}\right)\right)\right)H\left[P_r(1-\beta)-h'\right]e^{\theta L}}_{\text{Selling product with gift card to your customers}} \\
& + \underbrace{\left(a' + \left(P_r(1-\beta)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right]e^{\theta L}}_{\text{Selling gift card to third party}} \\
& \underbrace{\begin{aligned}
& + \underbrace{I_e L\left(\left[P_r(1-\beta)-h'\right](S-\lambda_1 Z)+\left[P_{rt}-h'\right]Z\right)}_{I} \\
& + \underbrace{\left(\frac{\left[P_r(1-\beta)-h'\right]\left((\alpha_1+\alpha_2)S-\lambda_1 Z\right)}{H} + \frac{\left[P_{rt}-h'\right](1-\lambda_3)Z}{H}\right) I_e \frac{H^2}{n_r^2}\sum_{i=1}^{n_r}\left((n_r - i)+\frac{1}{2}\right)e^{-i\theta\frac{H}{n_r}}}_{II} \\
& + \underbrace{\left(\frac{\left[P_r(1-\beta)-h'\right]\alpha_3 S + \left[P_{rt}-h'\right]\lambda_3 Z}{H}\right) I_e \frac{H^2}{n_r^2}\sum_{i=1}^{n_r} e^{-i\theta\frac{H}{n_r}}}_{III}
\end{aligned}}_{\text{The interest earned of selling gift cards before selling products}}
\end{aligned}
\tag{A.2}
$$

Appendix B: Calculation of average supplier inventory in model 1

If n_r is divisible by n_s, the average inventory of supplier for each period is equal and can be calculated as what follows.

First mode (divisible)

The average inventories for $\bar{I}_1_\bar{I}_{n_s}$periods in this mode are as below.

$$\bar{I}=\bar{I}_1=\ldots=\bar{I}_{n_s}=\frac{n_s}{H}\left[\left(\frac{n_r}{n_s}-1\right)D_sT_r\times T_r+\left(\frac{n_r}{n_s}-2\right)D_sT_r\times T_r+\ldots+D_sT_r\times T_r\right]$$
$$=\frac{n_s}{n_rT_r}\left[D_s{T_r}^2\left(\left(\frac{n_r}{n_s}-1\right)+\left(\frac{n_r}{n_s}-2\right)\ldots.+1\right)\right]=\frac{n_s}{n_rT_r}\left[D_s{T_r}^2\frac{\frac{n_r}{n_s}\left(\frac{n_r}{n_s}-1\right)}{2}\right]$$
$$=\frac{D_sT_r}{2}\left(\frac{n_r}{n_s}-1\right) \tag{B.1}$$

In the above formula, the total inventory level of each period is calculated and divided by the length of each period.

Now, if $n_r$cannot be divided byn_s, two cases occur, in which n_s/n_rmust be either rounded up or down. In fact, in the final periods of these modes, the supplier would replenish the retailer more or less than the previous periods, which leads to various average inventory values.

Second mode (rounding down)

The average of inventories for $\bar{I}_1$_$\bar{I}_{n_s}$periods in this mode are as follows.

$$\bar{I}_1=\ldots=\bar{I}_{n_{s-1}}=\frac{1}{\left[\frac{n_r}{n_s}\right]T_r}\left[\left(\left[\frac{n_r}{n_s}\right]-1\right)D_sT_r\times T_r+\left(\left[\frac{n_r}{n_s}\right]-2\right)D_sT_r\times T_r+\ldots+D_sT_r\times T_r\right]$$
$$=\frac{1}{\left[\frac{n_r}{n_s}\right]T_r}\left[D_s{T_r}^2\left(\left(\left[\frac{n_r}{n_s}\right]-1\right)+\left(\left[\frac{n_r}{n_s}\right]-2\right)\ldots.+1\right)\right]$$
$$=\frac{1}{\left[\frac{n_r}{n_s}\right]T_r}\left[D_s{T_r}^2\frac{\left[\frac{n_r}{n_s}\right]\left(\left[\frac{n_r}{n_s}\right]-1\right)}{2}\right]=\frac{D_sT_r}{2}\left(\left[\frac{n_r}{n_s}\right]-1\right) \tag{B.2}$$

$$I_{n_s}=\bar{I}_{n_s}e^{-\left(\theta(n_s-1)\left[\frac{n_r}{n_s}\right]T_r\right)}=\frac{(D_r+\alpha_2s)H}{2n_r}\left(n_r-\left[\frac{n_r}{n_s}\right](n_s-1)\right)e^{-\left(\theta(n_s-1)\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}\right)} \tag{B.3}$$

Third mode (rounding up)

The average of inventories for $\bar{I}_1$_$\bar{I}_{n_s}$periods in this mode are as follows.

$$\bar{I}_1=\ldots=\bar{I}_{n_{s-1}}=\frac{1}{\left[\frac{n_r}{n_s}\right]T_r}\left[\left(\left[\frac{n_r}{n_s}\right]-1\right)D_sT_r\times T_r+\left(\left[\frac{n_r}{n_s}\right]-2\right)D_sT_r\times T_r+\ldots+D_sT_r\times T_r\right]$$
$$=\frac{1}{\left[\frac{n_r}{n_s}\right]T_r}\left[D_s{T_r}^2\left(\left(\left[\frac{n_r}{n_s}\right]-1\right)+\left(\left[\frac{n_r}{n_s}\right]-2\right)\ldots.+1\right)\right]$$
$$=\frac{1}{\left[\frac{n_r}{n_s}\right]T_r}\left[D_s{T_r}^2\frac{\left[\frac{n_r}{n_s}\right]\left(\left[\frac{n_r}{n_s}\right]-1\right)}{2}\right]=\frac{D_sT_r}{2}\left(\left[\frac{n_r}{n_s}\right]-1\right) \tag{B.4}$$

$$I_{n_s}=\bar{I}_{n_s}e^{-\left(\theta(n_s-1)\left[\frac{n_r}{n_s}\right]T_r\right)}=\frac{(D_r+\alpha_2s)H}{2n_r}\left(n_r-\left[\frac{n_r}{n_s}\right](n_s-1)\right)e^{-\left(\theta(n_s-1)\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}\right)} \tag{B.5}$$

Appendix C. The profit function of whole supply chain

If n_r is divisible by n_s, echelon 2's profit function is:

$$PC_{rt21} = P_s\left(D_r + \alpha_2 s + \lambda_2 z\right)\frac{H}{n_s}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_s}}-1}\right] - A_s\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{T_r}{n_s}}-1}\right] - F\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]$$

$$-C_s\left(D_r + \alpha_2 s + \lambda_2 z\right)\frac{H}{n_s}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_s}}-1}\right] - I_h C_s\frac{\left(D_r + \alpha_2 s + \lambda_2 z\right)H}{2n_r}\left(\frac{n_r}{n_s}-1\right)\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_s}}-1}\right]$$

$$\underbrace{+P_r D_r\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right] - P_r\alpha_1\left(a+\beta b\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{selling product directly}} + \underbrace{\left(a+\beta b\right)H\left[P_r\left(1-\beta\right)-h'\right]e^{\theta L}}_{\text{selling product with gift cardto your customers}}$$

$$+\underbrace{\left(a' + \left(P_r\left(1-\beta\right)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right]e^{\theta L}}_{\text{selling gift card to third party}} + \underbrace{I_e L\left(\left[P\left(1-\beta\right)-h'\right]s + \left[P_{rt}-h'\right]z\right)H}_{\text{the interest earnedof selling gift cards before selling products }(I)}$$

$$+\underbrace{\begin{pmatrix}\left[P_r\left(1-\beta\right)-h'\right]\left(\alpha_1+\alpha_2\right)s \\ +\left[P_{rt}-h'\right]\left(1-\lambda_1\right)z\end{pmatrix} I_e\frac{H^2}{n_r^{\,2}}\sum_{i=1}^{n_r}\left(\left(n_r-i\right)+\frac{1}{2}\right)e^{-i\theta\frac{H}{n_r}}}_{\text{the interest earnedof selling gift cards before selling products }(II)} \tag{C.1}$$

$$+\underbrace{\left(\left[P_r\left(1-\beta\right)-h'\right]\alpha_3 s + \left[P_{rt}-h'\right]\lambda_3 z\right)I_e\frac{H^2}{n_r^{\,2}}\sum_{i=1}^{n_r}e^{-i\theta\frac{H}{n_r}}}_{\text{the interest earnedof selling gift cards before selling products }(III)}$$

$$-\left[\begin{array}{l}\underbrace{A_r\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{Ordering cost}} + \underbrace{I_{hr}P_s\frac{\left(D_r+\alpha_2\left(a+\beta b\right)+\left(a'+\left(P_r\left(1-\beta\right)-P_{tc}\right)b'\right)\right)T_r^{\,2}}{2}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{holding cost}} \\ \underbrace{+P_s\left(D_r+\alpha_2\left(a+\beta b\right)+\left(a'+\left(P_r\left(1-\beta\right)-P_{tc}\right)b'\right)\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{n_s H}{n_r}}-1}\right]}_{\text{Purchesing cost}}\end{array}\right]$$

$$+\underbrace{\left(a'+\left(P_r\left(1-\beta\right)-P_{tc}\right)b'\right)\left(P_{tc}-P_{rt}\right)}_{\text{third party's Income from saling gift card}}$$

With respect to incomes and calculated costs, the profit function of the whole supply chain is:

$$PC_{rt21}(\beta, P_{rt}) = \psi_2\beta + \psi_3\beta^2 + \psi_4 P_{rt} + \psi_6 P_{rt}^2 + \psi_5\beta P_{rt} + \psi_1 \tag{C.2}$$

Where

$$\psi_1 = \begin{pmatrix} \left(P_r D_r \frac{H}{n_r} - A_r - F - \frac{P_r \alpha_1 a}{n_r}\right)\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right] \\ +\left(aP_r - ah^{'}\right)\left(e^{\theta L} + I_e L + \frac{(\alpha_1+\alpha_2)V}{H} + \frac{\alpha_3}{H}W\right) \\ +\left(a^{'}h^{'} + b^{'}h^{'}P_r\right)\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right) \\ -ED_r - E\frac{\alpha_2}{H}\left(a+a^{'}\right) - E\frac{\lambda_2}{2H}P_r b^{'} - A_s\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{T_r}{n_s}}-1}\right] \\ +\left(D_r + \alpha_2 a + \frac{\lambda_2}{2}a^{'} + \frac{\lambda_2}{2}b^{'}P_r\right)Y \end{pmatrix}$$

$$\psi_2 = \begin{pmatrix} \left(-P_r\alpha_1 b\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]\right) + (P_r b - aP_r - b)\left(e^{\theta L} + I_e L + \frac{(\alpha_1+\alpha_2)V}{H} + \frac{\alpha_3}{H}W\right) \\ +b^{'}P_r h^{'}\left(e^{\theta L} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right) - E\frac{\alpha_2 b}{H} + E\frac{\lambda_2 b^{'}P_r}{2H} + \left(\alpha_2 b - \frac{\lambda_2}{2}b^{'}P_r\right)Y \end{pmatrix} \tag{C.3}$$

$$\psi_3 = -bP_r\left(e^{\theta L} + I_e L + \frac{(\alpha_1+\alpha_2)V}{H} + \frac{\alpha_3}{H}W\right)$$

$$\psi_4 = \left(a^{'} + b^{'}P_r + b^{'}h^{'}\right)\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right) + E\frac{\lambda_2 b^{'}}{2H} - \left(\frac{\lambda_2}{2}b^{'}\right)Y$$

$$\psi_5 = -b^{'}P_r\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right)$$

$$\psi_6 = -b^{'}\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right)$$

If n_r is not divisible by n_s and $\left[\frac{\boldsymbol{n_r}}{\boldsymbol{n_s}}\right]$ is trend downwards.

$$
\begin{aligned}
PC_{rt22} &= P_s\left(D_r+\alpha_2 s+\lambda_2 z\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_s}}-1}\right]-A_s\left[\frac{e^{-n_s\theta\left[\frac{n_r}{n_s}\right]T_r}-1}{e^{-\theta\left[\frac{n_r}{n_s}\right]T_r}-1}\right]-F\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]\\
&-C_s\left(D_r+\alpha_2 s+\lambda_2 z\right)\frac{H}{n_r}\left(\left[\frac{n_r}{n_s}\right]\left[\frac{e^{-(n_s-1)\theta\left[\frac{n_r}{n_s}\right]T_r}-1}{e^{-\theta\left[\frac{n_r}{n_s}\right]T_r}-1}\right]+e^{-\left((n_s-1)\theta\left[\frac{n_r}{n_s}\right]T_r\right)}\left(n_r-\left[\frac{n_r}{n_s}\right](n_s-1)\right)\right)\\
&-I_h C_s\frac{(D_r+\alpha_2 s+\lambda_2 z)H}{2n_r}\left(\left(\left[\frac{n_r}{n_s}\right]-1\right)\left[\frac{e^{-(n_s-1)\theta\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}}-1}{e^{-\theta\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}}-1}\right]+\left(n_r-\left[\frac{n_r}{n_s}\right](n_s-1)\right)e^{-\left(\theta(n_s-1)\left[\frac{n_r}{n_s}\right]\frac{H}{n_r}\right)}\right)\\
&\underbrace{+P_r D_r\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]-P_r\alpha_1(a+\beta b)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{selling product directly}}+\underbrace{(a+\beta b)H\left[P_r(1-\beta)-h'\right]e^{\theta L}}_{\text{selling product with gift cardto your customers}}\\
&+\underbrace{\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right]e^{\theta L}}_{\text{selling gift card to third party}}+\underbrace{I_e L\left(\left[P(1-\beta)-h'\right]s+\left[P_{rt}-h'\right]z\right)H}_{\text{the interest earnedof selling gift cards before selling products}(I)}\\
&+\underbrace{\begin{pmatrix}\left[P_r(1-\beta)-h'\right](\alpha_1+\alpha_2)s\\+\left[P_{rt}-h'\right](1-\lambda_1)z\end{pmatrix}I_e\frac{H^2}{n_r^2}\sum_{i=1}^{n_r}\left((n_r-i)+\frac{1}{2}\right)e^{-i\theta\frac{H}{n_r}}}_{\text{the interest earnedof selling gift cards before selling products }(II)}\\
&+\underbrace{\left(\left[P_r(1-\beta)-h'\right]\alpha_3 s+\left[P_{rt}-h'\right]\lambda_3 z\right)I_e\frac{H^2}{n_r^2}\sum_{i=1}^{n_r}e^{-i\theta\frac{H}{n_r}}}_{\text{the interest earnedof selling gift cards before selling products}(III)}\\
&-\left[\begin{gathered}\underbrace{A_r\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{Ordering cost}}+\underbrace{I_{hr}P_s\frac{\left(D_r+\alpha_2(a+\beta b)+\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\right)T_r^2}{2}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{holding cost}}\\
\underbrace{+P_s\left(D_r+\alpha_2(a+\beta b)+\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{n_s H}{n_r}}-1}\right]}_{\text{Purche sin g cost}}\end{gathered}\right]\\
&+\underbrace{\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\left(P_{tc}-P_{rt}\right)}_{\text{third party's Income from saling gift card}}
\end{aligned}
\tag{C.4}
$$

According to income and obtained costs, the profit function of whole supply chain is:

$$TP'_{rt22}(\beta,P_{rt})=\psi'_2\beta+\psi'_3\beta^2+\psi'_4P_{rt}+\psi'_6P_{rt}^2+\psi'_5\beta P_{rt}+\psi'_1 \tag{C.5}$$

Where

$$\psi_1' = \begin{pmatrix} \left(P_r D_r \frac{H}{n_r} - A_r - F - \frac{P_r\alpha_1 a}{n_r}\right)\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right] \\ +\left(aP_r - ah'\right)\left(e^{\theta L} + I_e L + \frac{(\alpha_1+\alpha_2)V}{H} + \frac{\alpha_3}{H}W\right) \\ +\left(a'h' + b'h'P_r\right)\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right) \\ -ED_r - E\frac{\alpha_2}{H}\left(a+a'\right) - E\frac{\lambda_2}{2H}P_r b' - A_s\left[\frac{e^{-n_s\theta\left[\frac{n_r}{n_s}\right]T_r}-1}{e^{-\theta\left[\frac{n_r}{n_s}\right]T_r}-1}\right] \\ +\left(D_r + \alpha_2 a + \frac{\lambda_2}{2}a' + \frac{\lambda_2}{2}b'P_r\right)Y \end{pmatrix}$$

$$\psi_2' = \begin{pmatrix} \left(-P_r\alpha_1 b\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]\right) + (P_r b - aP_r - b)\left(e^{\theta L} + I_e L + \frac{(\alpha_1+\alpha_2)V}{H} + \frac{\alpha_3}{H}W\right) \\ +b'P_r h'\left(e^{\theta L} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right) - E\frac{\alpha_2 b}{H} + E\frac{\lambda_2 b'P_r}{2H} + \left(\alpha_2 b - \frac{\lambda_2}{2}b'P_r\right)Y \end{pmatrix} \tag{C.6}$$

$$\psi_3' = -bP_r\left(e^{\theta L} + I_e L + \frac{(\alpha_1+\alpha_2)V}{H} + \frac{\alpha_3}{H}W\right)$$

$$\psi_4' = \left(a' + b'P_r + b'h'\right)\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right) + E\frac{\lambda_2 b'}{2H} - \left(\frac{\lambda_2}{2}b'\right)Y$$

$$\psi_5' = -b'P_r\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right)$$

$$\psi_6' = -b'\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H}W\right)$$

If n_r is not divisible by n_s and $\left[\frac{\boldsymbol{n_r}}{\boldsymbol{n_s}}\right]$ is trend upwards.

$$
\begin{aligned}
PC_{rt23} &= P_s\left(D_r+\alpha_2 s+\lambda_2 z\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_s}}-1}\right]-A_s\left[\frac{e^{-n_s\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r}-1}{e^{-\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r}-1}\right]-F\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]\\
&-C_s\left(D_r+\alpha_2 s+\lambda_2 z\right)\frac{H}{n_r}\left(\left(\left[\frac{n_r}{n_s}\right]+1\right)\left[\frac{e^{-(n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r}-1}{e^{-\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r}-1}\right]+e^{-\left((n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r\right)}\left(n_r-\left(\left[\frac{n_r}{n_s}\right]+1\right)(n_s-1)\right)\right)\\
&-I_h C_s\frac{\left(D_r+\alpha_2 s+\lambda_2 z\right)H}{2n_r}\left(\left[\frac{n_r}{n_s}\right]\left[\frac{e^{-(n_s-1)\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)\frac{H}{n_r}}-1}{e^{-\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)\frac{H}{n_r}}-1}\right]+\left(n_r-\left(\left[\frac{n_r}{n_s}\right]+1\right)(n_s-1)\right)e^{-\left(\theta(n_s-1)\left(\left[\frac{n_r}{n_s}\right]+1\right)\frac{H}{n_r}\right)}\right)\\
&\underbrace{+P_r D_r\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]-P_r\alpha_1(a+\beta b)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{selling product directly}}+\underbrace{(a+\beta b)H\left[P_r(1-\beta)-h'\right]e^{\theta L}}_{\text{selling product with gift card to your customers}}\\
&+\underbrace{\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right]e^{\theta L}}_{\text{selling gift card to third party}}+\underbrace{I_e L\left(\left[P(1-\beta)-h'\right]s+\left[P_{rt}-h'\right]z\right)H}_{\text{the interest earned of selling gift cards before selling products (I)}}\\
&+\underbrace{\begin{pmatrix}\left[P_r(1-\beta)-h'\right](\alpha_1+\alpha_2)s\\+\left[P_{rt}-h'\right](1-\lambda_1)z\end{pmatrix}I_e\frac{H^2}{n_r^2}\sum_{i=1}^{n_r}\left((n_r-i)+\frac{1}{2}\right)e^{-i\theta\frac{H}{n_r}}}_{\text{the interest earned of selling gift cards before selling products (II)}}\\
&+\underbrace{\left(\left[P_r(1-\beta)-h'\right]\alpha_3 s+\left[P_{rt}-h'\right]\lambda_3 z\right)I_e\frac{H^2}{n_r^2}\sum_{i=1}^{n_r}e^{-i\theta\frac{H}{n_r}}}_{\text{the interest earned of selling gift cards before selling products (III)}}\\
&-\left[\begin{gathered}\underbrace{A_r\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{Ordering cost}}+\underbrace{I_h P_s\frac{\left(D_r+\alpha_2(a+\beta b)+\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\right)T_r^2}{2}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{\text{holding cost}}\\
\underbrace{+P_s\left(D_r+\alpha_2(a+\beta b)+\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{n_s H}{n_r}}-1}\right]}_{\text{Purchesing cost}}\end{gathered}\right]\\
&+\underbrace{\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\left(P_{tc}-P_{rt}\right)}_{\text{third party's Income from saling gift card}}
\end{aligned}
\tag{C.7}
$$

According to income and obtained costs, the profit function of whole supply chain is:

$$TP''_{rt23}(\beta,P_{rt})=\psi''_2\beta+\psi''_3\beta^2+\psi''_4 P_{rt}+\psi''_6 P_{rt}^2+\psi''_5\beta P_{rt}+\psi''_1 \tag{C.8}$$

Where

$$\psi_1'' = \begin{pmatrix} \left(P_r D_r \frac{H}{n_r} - A_r - \frac{P_r \alpha_1 a}{n_r} - F\right)\left[\frac{e^{-\theta H} - 1}{e^{-\theta \frac{H}{n_r}} - 1}\right] \\ +(aP_r - ah')\left(e^{\theta L} + I_e L + \frac{(\alpha_1 + \alpha_2)V}{H} + \frac{\alpha_3}{H} W\right) \\ +(a'h' + b'h'P_r)\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H} W\right) \\ -ED_r - E\frac{\alpha_2}{H}(a + a') - E\frac{\lambda_2}{2H} P_r b' - A_s \left[\frac{e^{-n_s \theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r} - 1}{e^{-\theta\left(\left[\frac{n_r}{n_s}\right]+1\right)T_r} - 1}\right] \\ +\left(D_r + \alpha_2 a + \frac{\lambda_2}{2} a' + \frac{\lambda_2}{2} b' P_r\right) Y \end{pmatrix}$$

$$\psi_2'' = \begin{pmatrix} \left(-P_r \alpha_1 b \frac{H}{n_r}\left[\frac{e^{-\theta H} - 1}{e^{-\theta \frac{H}{n_r}} - 1}\right]\right) + (P_r b - aP_r - b)\left(e^{\theta L} + I_e L + \frac{(\alpha_1 + \alpha_2)V}{H} + \frac{\alpha_3}{H} W\right) \\ +b'P_r h'\left(e^{\theta L} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H} W\right) - E\frac{\alpha_2 b}{H} + E\frac{\lambda_2 b' P_r}{2H} + \left(\alpha_2 b - \frac{\lambda_2}{2} b' P_r\right) Y \end{pmatrix} \tag{C.9}$$

$$\psi_3'' = -bP_r\left(e^{\theta L} + I_e L + \frac{(\alpha_1 + \alpha_2)V}{H} + \frac{\alpha_3}{H} W\right)$$

$$\psi_4'' = (a' + b'P_r + b'h')\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H} W\right) + E\frac{\lambda_2 b'}{2H} - \left(\frac{\lambda_2}{2} b'\right) Y$$

$$\psi_5'' = -b'P_r\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H} W\right)$$

$$\psi_6'' = -b'\left(\frac{e^{\theta L}}{2} + I_e L + \frac{(1-\lambda_3)V}{H} + \frac{\lambda_3}{H} W\right)$$

So, the profit function of the whole supply chain is

$$PC_{rt2}(n_s, n_r, \beta, P_{rt}, P_{tc}) = \begin{cases} PC_{rt21}(n_s, n_r, \beta, P_{rt}, P_{tc}) if \frac{n_r}{n_s} = \left[\frac{n_r}{n_s}\right] \\ PC_{rt22}(n_s, n_r, \beta, P_{rt}, P_{tc}) if \frac{n_r}{n_s} \neq \left[\frac{n_r}{n_s}\right] RoundDown \left[\frac{n_r}{n_s}\right] \\ PC_{rt23}(n_s, n_r, \beta, P_{rt}, P_{tc}) if \frac{n_r}{n_s} \neq \left[\frac{n_r}{n_s}\right] \quad RoundUp \left[\frac{n_r}{n_s}\right] \end{cases} \tag{C.10}$$

Appendix D: Developing the profit retailer

According to costs and revenues of the retailer in Section 3.1.2, the retailer's profit is equal to:

$$
\begin{aligned}
PC_r ={}& \underbrace{P_r D_r \frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right] - P_r\alpha_1\frac{(a+\beta b)}{H}\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{selling\ product\ directly} \\
&+\underbrace{\left[(a+\beta b)-\lambda_1\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\right]H\left[P_r(1-\beta)-h'\right]e^{\theta L}}_{selling\ product\ with\ gift\ card\ to\ your\ customers} \\
&+\underbrace{\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right]e^{\theta L}}_{selling\ gift\ card\ to\ third\ party} \\
&+\underbrace{I_e L\left((a+\beta b)\left[P_r(1-\beta)-h'\right]+\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\left[P_{rt}-h'\right]\right)H}_{the\ interest\ earned\ of\ selling\ gift\ cards\ before\ selling\ products\ (I)} \\
&+\underbrace{\begin{pmatrix}(a+\beta b)H\left[P_r(1-\beta)-h'\right](\alpha_1+\alpha_2)\\ +\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right](1-\lambda_1)\end{pmatrix} I_e\frac{H^2}{n_r^2}\sum_{i=1}^{n_r}\left((n_r-i)+\frac{1}{2}\right)e^{-i\theta\frac{H}{n_r}}}_{the\ interest\ earned\ of\ selling\ gift\ cards\ before\ selling\ products\ (II)} \\
&+\underbrace{\begin{pmatrix}(a+\beta b)H\left[P_r(1-\beta)-h'\right]\alpha_3\\ +\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)H\left[P_{rt}-h'\right]\lambda_3\end{pmatrix} I_e\frac{H^2}{n_r^2}\sum_{i=1}^{n_r}e^{-i\theta\frac{H}{n_r}}}_{the\ interest\ earned\ of\ selling\ gift\ cards\ before\ selling\ products\ (III)} \\
&-\left[\begin{aligned}&\underbrace{A_r\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{Ordering\ cost}+\underbrace{I_{hr}P_s\frac{\left(D_r+\alpha_2(a+\beta b)+\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\right)T_r^2}{2}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]}_{holding\ cost}\\
&+\underbrace{P_s\left(D_r+\alpha_2(a+\beta b)+\left(a'+\left(P_r(1-\beta)-P_{tc}\right)b'\right)\right)\frac{H}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{n_s H}{n_r}}-1}\right]}_{Purchesing\ cost}\end{aligned}\right]
\end{aligned}
\tag{D.1}
$$

Eq. D.1 can be written as below.

$$
\begin{aligned}
PC_r(\beta,P_{rt}) ={}& \left(-P_r\alpha_1\frac{b}{n_r}\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]+\left((P_r-h')b-P_r a\right)HE_4+\frac{b'P_r}{2}Hh'E_5-\alpha_2 bE_3\right. \\
&\left.+\frac{b'P_r}{2}\lambda_2E_3\right)\beta \\
&-bP_rHE_4\beta^2+\left(\left(\frac{b'P_r+a'+h'b'}{2}\right)E_5+\lambda_2\frac{b'P_{rt}}{2}E_3\right)P_{rt}-\frac{b'}{2}HE_5P_{rt}^2-\frac{b'P_r}{2}HE_5\beta P_{rt} \\
&+\begin{pmatrix}\left(P_rD_r\frac{H}{n_r}-P_r\alpha_1\frac{a}{n_r}-A_r\right)\left[\frac{e^{-\theta H}-1}{e^{-\theta\frac{H}{n_r}}-1}\right]+(P_r a-h'a)HE_4-\left(\frac{a'h'}{2}+\frac{b'h'P_r}{2}\right)HE_5\\ -\left(D_r+\alpha_2 a+\lambda_2\frac{b'P_r}{2}+\lambda_2 a'-\frac{a'}{2}\right)E_3\end{pmatrix}
\end{aligned}
\tag{D.2}
$$

Where

$$E_1 = I_e \frac{H^2}{{n_r}^2} \sum_{i=1}^{n_r} \left((n_r - i) + \frac{1}{2}\right) e^{-i\theta \frac{H}{n_r}}$$
$$E_2 = I_e \frac{H^2}{{n_r}^2} \sum_{i=1}^{n_r} e^{-i\theta \frac{H}{n_r}}$$
$$E_3 = \left[I_{hr} P_s \frac{{T_r}^2}{2} \left[\frac{e^{-\theta H} - 1}{e^{-\theta \frac{H}{n_r}} - 1} \right] + P_s \frac{H}{n_r} \left[\frac{e^{-\theta H} - 1}{e^{-\theta \frac{n_s H}{n_r}} - 1} \right] \right] \tag{D.3}$$
$$E_4 = \left(e^{\theta L} + I_e L + E_1(\alpha_1 + \alpha_2) + \alpha_3 E_2\right)$$
$$E_5 = \left(e^{\theta L} + I_e L + (1 - \lambda_1) E_1 + E_2 \lambda_3\right)$$

Finally, we have:

$$PC_r(\beta, P_{rt}) = \psi_2 \beta + \psi_3 \beta^2 + \psi_4 P_{rt} + \psi_6 P_{rt}^2 + \psi_5 \beta P_{rt} + \psi_1 \tag{D.4}$$

Where

$$\psi_1 = \begin{pmatrix} \left(P_r D_r \frac{H}{n_r} - P_r \alpha_1 \frac{a}{n_r} - A_r\right) \left[\frac{e^{-\theta H} - 1}{e^{-\theta \frac{H}{n_r}} - 1} \right] + \left(P_r a - h^{'} a\right) H E_4 - \left(\frac{a^{'} h^{'}}{2} + \frac{b^{'} h^{'} P_r}{2}\right) H E_5 \\ -\left(D_r + \alpha_2 a + \lambda_2 \frac{b^{'} P_r}{2} + \lambda_2 a^{'} - \frac{a^{'}}{2}\right) E_3 \end{pmatrix}$$
$$\psi_2 = -P_r \alpha_1 \frac{b}{n_r} \left[\frac{e^{-\theta H} - 1}{e^{-\theta \frac{H}{n_r}} - 1} \right] + \left((P_r - h^{'}) b - P_r a\right) H E_4 + \frac{b^{'} P_r}{2} H h^{'} E_5 - \alpha_2 b E_3 + \frac{b^{'} P_r}{2} \lambda_2 E_3 \tag{D.5}$$
$$\psi_3 = -b P_r H E_4$$
$$\psi_4 = \left(\frac{b^{'} P_r + a^{'} + h^{'} b^{'}}{2}\right) E_5 + \lambda_2 \frac{b^{'} P_{rt}}{2} E_3$$
$$\psi_5 = -\frac{b^{'} P_r}{2} H E_5$$
$$\psi_6 = -\frac{b^{'}}{2} H E_5$$

Appendix E: Proofing concavity of $PC_r(\beta, n_r, P_{rt})$

$PC_r(\beta, n_r, P_{rt})$is concave if and only if $XHX^T < 0$ where $X = [\beta, P_{rt}]$, $X^T = \begin{bmatrix} \beta \\ P_{rt} \end{bmatrix}$ and $H = \begin{bmatrix} \frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial^2 \beta} & \frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial \beta \partial P_{rt}} \\ \frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial P_{rt} \partial \beta} & \frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial^2 P_{rt}} \end{bmatrix}$

Now according to the expressions X, X^T,and H, the value of the phrase $XHX^T$is equal to$\left[\beta\beta \frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial^2 \beta} + \beta P_{rt} \frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial P_{rt} \partial \beta} + P_{rt} \beta \frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial \beta \partial P_{rt}} + P_{rt} P_{rt} \frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial^2 P_{rt}}\right]$

Where

$$PC_r(\beta, P_{rt}) = \psi_2 \beta + \psi_3 \beta^2 + \psi_4 P_{rt} + \psi_6 P_{rt}^2 + \psi_5 \beta P_{rt} + \psi_1$$
$$\beta\beta \frac{\partial^2 PC_r(\beta, n_r, P_{rt})}{\partial^2 \beta} = 2\psi_3 \beta\beta$$

$$\beta P_{rt}\frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial P_{rt}\partial\beta}=\psi_5\beta P_{rt}$$
$$P_{rt}\beta\frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial\beta\partial P_{rt}}=+\psi_5$$
$$P_{rt}P_{rt}\frac{\partial^2 PC_r(\beta,n_r,P_{rt})}{\partial^2 P_{rt}}=+2\psi_6$$

So, we have

$$\begin{aligned}XHX^T &= 2\beta\beta\psi_3+2\beta P_{rt}\psi_5+2P_{rt}P_{rt}\psi_6\\
&=-2bP_r\left(H+I_eLH+I_e\frac{H^2}{{n_r}^2}(\alpha_1+\alpha_2)E_1+\alpha_3E_2\right)\beta\beta\\
&-\beta P_{rt}b'\left(P_rHe^{\theta L}+I_eLHP_r+I_e\frac{H^2}{{n_r}^2}\lambda_1E_1P_r+E_2(1-\lambda_1)P_r\right)\beta P_{rt}\\
&-P_{rt}P_{rt}\psi_6b'\left(He^{\theta L}+I_eLH+I_e\frac{H^2}{{n_r}^2}\lambda_1E_1+E_2(1-\lambda_1)\right)P_{rt}^2\end{aligned}$$

So the total profit of $PC_r(\beta,n_r,P_{rt})$is concave.

Appendix F: Finding the roots of $PC(\beta,P_{rt})$ accorrding to $\beta$and P_{rt}

From Eq. 26, we have:

$$PC(\beta,P_{rt})=\psi_2\beta+\psi_3\beta^2+\psi_4P_{rt}+\psi_6P_{rt}^2+\psi_5\beta P_{rt}+\psi_1 \tag{F.1}$$

Taking the first derivatives of $PC(\beta,P_{rt})$ with respect to $\beta$and P_{rt} gives:

$$\frac{PC(\beta,P_{rt})}{d\beta}=\psi_2+2\psi_3\beta+\psi_5P_{rt}\rightarrow\beta=-\frac{\psi_2+\psi_5P_{rt}}{2\psi_3} \tag{F.2}$$
$$\frac{PC(\beta,P_{rt})}{dP_{rt}}=\psi_4+2\psi_6P_{rt}+\psi_5\beta\rightarrow P_{rt}=-\frac{\psi_4+\psi_5\beta}{2\psi_6} \tag{F.3}$$

And after some algebra we have:

$$\beta=\frac{\psi_2 2\psi_6-\psi_4\psi_5}{\psi_5\psi_5-2\psi_6 2\psi_3} \tag{F.4}$$
$$P_{rt}=-\frac{\psi_4+\psi_5\beta}{2\psi_6} \tag{F.5}$$

Appendix G: Proving $\beta<1$

$P_{rt}>P_s$is a necessary and logical condition. Now according to Eq. 30 we have:

$$-\frac{\psi_4+\psi_5\beta}{2\psi_6}\geq P_s\overset{\psi_6<0}{\Rightarrow}-\psi_4-\psi_5\beta<2\psi_6P_s \tag{G.1}$$

Now according to mathematical calculations and simplification we have

$$-\psi_5\beta<2\psi_6P_s+\psi_4\overset{-\psi_5\geq0}{\Rightarrow}\beta<\frac{2\psi_6P_s+\psi_4}{-\psi_5} \tag{G.2}$$

Now according to the $\psi_6, \psi_4, \psi_5, X, E$ we have

$$\beta < \frac{-2b'HXP_s + \left(a' + b'P_r + b'h' + b'\lambda_1(P_r - h')\right)HX + E\lambda_2 b'}{b'P_r H(1-\lambda_1)X}$$

$$\frac{-2b'HXP_s + \left(a' + b'P_r + b'h' + b'\lambda_1(P_r - h')\right)HX + E\lambda_2 b'}{b'P_r H(1-\lambda_1)X} < 1 \xrightarrow{X>1} \tag{G.3}$$

$$-\left(2P_s + P_r(1-\lambda_1)\right)b'HX + \left(a' + b'P_r + b'h' + b'\lambda_1(P_r - h')\right)HX + E\lambda_2 b' < 0$$

$$-2b'P_s - 2b'P_r\lambda_1 - \left(a' + b'h'\right)(1-\lambda_1) - E\lambda_2 b' < 0$$